4.12. Example

Consider the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ from Example 1.2, where $\mathcal{A} = C^\infty(\mathbb{T})$, $\mathcal{H} = L^2(\mathbb{T}, m)$, and $\mathcal{D}$ the closure of

$$\mathcal{D}_0 : \mathcal{H} \supseteq \operatorname{dom} \mathcal{D}_0 \to \mathcal{H}, \quad g \mapsto \frac{1}{i} g'$$

with $\operatorname{dom} \mathcal{D}_0 = C^1(\mathbb{T})$. Then $\Delta := \mathcal{D}^2$ has spectrum

$$Sp(\Delta) = \{ |n|^2 \mid n \in \mathbb{Z} \}$$

We conclude that $(1+\Delta)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, i.e., the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $(1,\infty)$-summable.

Indeed, since the Fourier transform $\mathcal{F} : L^2(\mathbb{T}, m) \to \ell^2(\mathbb{Z})$ is a unitary and $\mathcal{F}\mathcal{D}\mathcal{F}^{-1} = M_{(n)_{n\in\mathbb{Z}}}$, where M_λ, for any sequence $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, is the closed operator

$$M_\lambda : \ell^2(\mathbb{Z}) \supseteq \operatorname{dom} M_\lambda \to \ell^2(\mathbb{Z}), \quad (a_n)_{n\in\mathbb{Z}} \mapsto (\lambda_n a_n)_{n\in\mathbb{Z}}$$

with domain $\operatorname{dom} M_\lambda := \{ (a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \mid (\lambda_n a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \}$, we conclude that

$$\mathcal{F}\Delta\mathcal{F}^{-1} = M_{(|n|^2)_{n\in\mathbb{Z}}} \quad \text{and} \quad \mathcal{F}(1+\Delta)^{-\frac{1}{2}}\mathcal{F}^{-1} = \overbrace{M_{\left(\frac{1}{\sqrt{1+n^2}}\right)_{n\in\mathbb{Z}}}}^{\in B(\ell^2(\mathbb{Z}))}$$

and finally $(1+\Delta)^{-\frac{1}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$; in fact, we have

$$\left(\mu_n\left((1+\Delta)^{-1/2}\right)\right)_{n=0}^\infty = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{1+n^2}}, \frac{1}{\sqrt{1+n^2}}, \ldots\right),$$

so that $\gamma_N\left((1+\Delta)^{-\frac{1}{2}}\right) \to 2$ as $N \to \infty$ and hence even $(1+\Delta)^{-\frac{1}{2}} \in \mathcal{M}(\mathcal{H})$ with $⨍ (1+\Delta)^{-\frac{1}{2}} = 2$.

4.13. Example:

In the situation of Example 4.12, let $P \in B(\mathcal{H})$ be the orthogonal projection onto $\ker \Delta = \mathbb{C}1 \subset L^2(\mathbb{T}, m)$.
By Exercise 1B-1, $(\mathcal{A}, \mathcal{H}, \tilde{D})$ with $\tilde{D} := D_P = D + P$ gives another spectral triple. Note that $\tilde{\Delta} := \tilde{D}^2 = \Delta + P$, so that $\tilde{\Delta}$ is invertible. We have $\tilde{\Delta}^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, since

$$\left(\mu_n(\tilde{\Delta}^{-1/2})\right)_{n=0}^{\infty} = \left(1, 1, 1, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n}, \dots\right)$$

and thus $\gamma_N(\tilde{\Delta}^{-\frac{1}{2}}) \longrightarrow 2$ as $N \to \infty$; in fact, we have $\tilde{\Delta}^{-\frac{1}{2}} \in \mathcal{M}(\mathcal{H})$ and $⨍ \tilde{\Delta}^{-\frac{1}{2}} = 2$.

More generally, for $f \in \mathcal{A} = C^\infty(\mathbb{T})$, we have $f \cdot \tilde{\Delta}^{-\frac{1}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, because $f\cdot : L^2(\mathbb{T}, m) \to L^2(\mathbb{T}, m)$, $g \mapsto fg$ is bounded and $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$.

It is a consequence of Connes' trace theorem that $f \cdot \tilde{\Delta}^{-\frac{1}{2}}$ is measurable and

$$⨍ f \cdot \tilde{\Delta}^{-1/2} = \frac{1}{\pi} \int_{\mathbb{T}} f(\zeta)\, dm(\zeta). \tag{4.5}$$

The general version of Connes' trace theorem (1988) is about pseudodifferential operators on compact Riemannian manifolds. This theory has its origins in the work of Kohn, Nirenberg, Hörmander and others in the 60's.

4.14. Definition

Let $\mathcal{M}$ be a compact smooth manifold of dimension n and let $\pi_E : E \to \mathcal{M}$ be a k-dimensional smooth vector bundle.

(i) A <u>differential operator of order m</u> is a linear operator

$$P : \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, E)$$

which, in local coordinates $x = (x_1, \ldots, x_n)$ of $\mathcal{M}$, is of the form

$$P = \sum_{|\alpha| \le m} A_\alpha(x) \, (-i)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with $0 \le \alpha_1, \ldots, \alpha_n \le n$ with cardinality $|\alpha| := \sum_{j=1}^n \alpha_j$ and $A_\alpha \in M_k(C^\infty(\mathcal{M}))$ for each $|\alpha| \le m$ with $A_\alpha \not\equiv 0$ for some $|\alpha| = m$.

(ii) For $\xi \in T_x^*\mathcal{M}$, written as $\xi = \sum_{j=1}^n \xi_j \, dx_j$, we define the <u>complete symbol of P</u> (as the polynomial in $\xi_1, \ldots, \xi_n$ given) by

$$p^P(x,\xi) := \sum_{d=0}^m p_d^P(x,\xi), \quad p_d^P(x,\xi) := \sum_{|\alpha| = d} A_\alpha(x) \, \underbrace{\xi^\alpha}_{= \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}.$$

The <u>principal symbol of P</u> is defined as

$$\sigma^P(x,\xi) := p_m^P(x,\xi) = \sum_{|\alpha| = m} A_\alpha(x) \, \xi^\alpha.$$

It induces a linear map $\sigma^P(\xi) : E_x \to E_x$ for each $\xi \in T_x^*\mathcal{M}$.

(iii) We say that the differential operator P is <u>elliptic</u>, if its principal symbol $\sigma^P(\xi): E_x \to E_x$ is invertible for each $x \in M$ and any $0 \neq \xi \in T_x^* M$.

(iv) For a local section u of E, one can write

$$(Pu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} p^P(x,\xi)\, \hat{u}(\xi)\, d\xi_1 \dots d\xi_n \quad (4.6)$$

with the <u>Fourier transform</u> $\hat{u}$ of u that is given by

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} u(x)\, dx_1 \dots dx_n .$$

A linear operator $P: \Gamma(M,E) \to \Gamma(M,E)$ is called <u>pseudodifferential operator of order m</u> ($m \in \mathbb{R}$), written $P \in \Psi^m(M;E)$, if (4.6) holds locally for a matrix-valued function p^P in the symbol class $\mathrm{Sym}^m(M;E)$, i.e., in local coordinates, a matrix of smooth functions whose derivatives satisfy the growth condition

$$|\partial_x^\alpha \partial_\xi^\beta p_{ij}^P(x,\xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\beta|} .$$

The <u>principal symbol</u> of P is then defined as

$$\sigma^P := [p^P] \in \mathrm{Sym}^m(M;E) / \mathrm{Sym}^{m-1}(M;E) .$$

(v) Suppose that g is a Riemannian metric on M. The <u>Wodzicki residue</u> of $P \in \Psi^{-n}(M;E)$ is defined by

$$\operatorname{Res}_W(P) := \frac{1}{(2\pi)^n} \int_{S^*\mathcal{M}} \operatorname{tr} \sigma^P(x,\xi)\, \omega_\xi \wedge dx$$

where $S^*\mathcal{M} := \{(x,\xi) \in T^*\mathcal{M} \mid \langle \xi,\xi \rangle_{T_x^*\mathcal{M}} = 1\}$ is the <u>co-sphere bundle</u> over $\mathcal{M}$, tr the matrix trace, and

$$dx := dx_1 \wedge \ldots \wedge dx_n,$$

$$\omega_\xi := \sum_{j=1}^{n} (-1)^{j-1} \xi_j\, d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n .$$

<u>4.15. Theorem</u> (Connes' trace theorem, 1988):

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension n. For $P \in \Psi^{-n}(\mathcal{M}; E)$ with a complex (!) vector bundle E over $\mathcal{M}$, the following statements hold:

(i) P extends to a bounded linear operator on the <u>Hilbert space</u> $L^2(\mathcal{M}, E)$, which is obtained by completion of $\Gamma(\mathcal{M}, E)$ with respect to the inner product given by

$$\langle u_1, u_2 \rangle := \int_{\mathcal{M}} \underbrace{\underbrace{u_2(x)^*}_{\in E_x^*} \underbrace{u_1(x)}_{\in E_x}}_{\in \mathbb{C}} \, d\mathrm{vol}(x) .$$

Moreover, $P \in \mathcal{L}^{(1,\infty)}(L^2(\mathcal{M}, E))$.

(ii) P is measurable and we have that $⨍ P = \frac{1}{n} \operatorname{Res}_W(P)$.

(iii) $⨍ P$ depends only on the conformal class of g on $\mathcal{M}$.