4. The Riemannian-Lebesgue measure in noncommutative geometry

In Chapter 3, we have seen that the geodesic distance on a connected, compact, and oriented Riemannian manifold can be recovered from its associated Hodge-de Rham triple via Connes' spectral distance.
In this chapter, we will discuss the noncommutative integral, by which the integration of (smooth) functions with respect to the Riemann-Lebesgue measure on Riemannian manifolds as introduced in Theorem 2.17 is generalized to the framework of spectral triples.

Like in quantum mechanics, the underlying idea is that operators on a separable complex Hilbert space H with $\dim H = \infty$ take over the role of complex variables, while selfadjoint operators correspond to real variables.

4.1. Remark:

Recall (Theorem 9.8, FA I) that $T \in B(\mathcal{H})$ is compact if and only if T can be approximated in operator norm on $B(\mathcal{H})$ by finite rank operators; equivalently,

$$\forall \varepsilon > 0 \ \exists V \subseteq H \text{ subspace}, \dim V < \infty \ : \ \| T|_{V^\perp} \| < \varepsilon,$$

where $T|_{V^\perp} : V^\perp \to \mathcal{H}$ is the restriction of T to $V^\perp$ and

$\|\cdot\|$ the norm on $B(V^\perp, \mathcal{H})$.

For a compact operator $T \in B(\mathcal{H})$, we call the non-zero eigenvalues $(\mu_n(T))_{n=0}^{\infty}$ of $|T| := (T^*T)^{1/2}$, arranged in decreasing order and repeated according to multiplicity, the <u>characteristic values of T</u>; note that $\mu_n(T) \to 0$ as $n \to \infty$. We have that for all $n \in \mathbb{N}_0$

$$\mu_n(T) = \inf\{\|T - S\| \mid S \in B(\mathcal{H}) : \dim \operatorname{ran} S \leq n\} \tag{4.1}$$
$$= \inf\{\|T|_{V^\perp}\| \mid V \subseteq \mathcal{H} \text{ subspace} : \dim V = n\},$$

and in particular $\mu_0(T) = \|T\|$.

In view of Remark 4.1, compact operators are considered as infinitesimals in our "quantized calculus"; their "size" is measured by the rate of decay of their sequence of characteristic values.

<u>4.2. Definition</u>:

Let $T \in K(\mathcal{H})$ and $\alpha > 0$ be given. We say that T is an <u>infinitesimal of order α</u> if

$$\mu_n(T) = O(n^{-\alpha}) \quad \text{as } n \to \infty,$$

i.e., if $\exists C < \infty \ \forall n \geq 1 : \mu_n(T) \leq C n^{-\alpha}$. For $\alpha > 0$, we denote by $I_\alpha(\mathcal{H})$ the set of all $T \in K(\mathcal{H})$ which are infinitesimals of order α.

4.3. Remark:

Recall (Theorem 9.5, FA I) that $K(\mathcal{H})$ is a norm-closed two-sided ideal in $B(\mathcal{H})$. It follows from (4.1) that

$$\mu_n(TS) \le \|S\|\,\mu_n(T) \quad \text{and} \quad \mu_n(ST) \le \|S\|\,\mu_n(T)$$

for all $T \in K(\mathcal{H})$ and $S \in B(\mathcal{H})$, and for $T_1, T_2 \in K(\mathcal{H})$,

$$\mu_{n+m}(T_1 + T_2) \le \mu_n(T_1) + \mu_m(T_2);$$

thus, each $I_\alpha(\mathcal{H})$ forms a (non-closed) two-sided ideal in $B(\mathcal{H})$.

Furthermore, we have for $T_1, T_2 \in K(\mathcal{H})$ that

$$\mu_{n+m}(T_1 T_2) \le \mu_n(T_1)\,\mu_m(T_2),$$

which implies the following rule for infinitesimals:

$$\left.\begin{array}{l} T_1 \text{ of order } \alpha_1 \\ T_2 \text{ of order } \alpha_2 \end{array}\right\} \Longrightarrow T_1 T_2 \text{ of order } \alpha_1 + \alpha_2 .$$

We want to find an "integral" that is defined on $I_1(\mathcal{H})$ and neglects all infinitesimals of order $\alpha > 1$.

4.4 Remark:

An operator $T \in B(\mathcal{H})$ is said to be in the trace class if

$$\sum_{k=0}^{\infty} \langle |T| \xi_k, \xi_k \rangle < \infty$$

for some (and hence for each) orthonormal basis $(\xi_k)_{k=0}^{\infty}$ of $\mathcal{H}$. In this case, the sum $\sum_{k=0}^{\infty} \langle T\xi_k, \xi_k \rangle$ is absolutely convergent and its value $\mathrm{Tr}(T)$ is independent of the choice of the orthonormal basis $(\xi_k)_{k=0}^{\infty}$ of $\mathcal{H}$; we call $\mathrm{Tr}(T)$ the <u>trace of T</u>.

The set $\mathcal{L}^1(\mathcal{H})$ of all trace class operators on $\mathcal{H}$ forms a (non-closed) two-sided ideal in $\mathcal{B}(\mathcal{H})$. However, $\mathcal{L}^1(\mathcal{H})$ is a Banach space with respect to the norm

$$\|T\|_1 := \mathrm{Tr}(|T|).$$

Note that $\mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$. If $T \in \mathcal{L}^1(\mathcal{H})$ is positive, then

$$\mathrm{Tr}(T) = \sum_{n=0}^{\infty} \mu_n(T). \tag{4.2}$$

Thus, $\mathrm{Tr}: \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ is not appropriate for our purpose, because:

- $\mathcal{I}_1(\mathcal{H})$ is not a subset of $\mathcal{L}^1(\mathcal{H})$, so that Tr is not defined on all infinitesimals of order 1;
- Tr does not vanish even on finite rank operators, but those belong to $\mathcal{I}_\alpha(\mathcal{H})$ for every $\alpha > 1$.

<u>4.5. Definition:</u>

For $T \in \mathcal{K}(\mathcal{H})$, we define for each $N \in \mathbb{N}$

$$\sigma_N(T) := \sum_{n=0}^{N-1} \mu_n(T) \quad \text{and} \quad \gamma_N(T) := \frac{1}{\log(N)} \sigma_N(T).$$

<u>4.6. Remark:</u>

Note that $\sigma_N(T)$ is a partial sum in (4.2). For $T \in \mathcal{I}_1(\mathcal{H})$, we find $C, C' < \infty$ such that $\sigma_N(T) \leq \|T\| + \sum_{n=1}^{N-1} \frac{C}{n} \leq C' \log(N)$ for all $N \in \mathbb{N}$; consequently, $(\gamma_N(T))_{N=1}^{\infty}$ is a bounded sequence. Thus $\mathcal{I}_1(\mathcal{H}) \subseteq \mathcal{L}^{(1,\infty)}(\mathcal{H})$, where

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) := \{ T \in K(\mathcal{H}) \mid \|T\|_{(1,\infty)} := \sup_{N \in \mathbb{N}} \gamma_N(T) < \infty \}$$

is the Dixmier ideal. Note that $\mathcal{L}^1(\mathcal{H}) \subseteq \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

4.7. Proposition:

Consider $T_1, T_2 \in K(\mathcal{H})$. For all $N \in \mathbb{N}$, we have that

$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2),$$

and if T_1, T_2 are positive, we have in addition that

$$\sigma_{2N}(T_1 + T_2) \geq \sigma_N(T_1) + \sigma_N(T_2).$$

It follows that for any positive $T_1, T_2 \in K(\mathcal{H})$

$$\gamma_N(T_1 + T_2) \leq \gamma_N(T_1) + \gamma_N(T_2) \leq \gamma_{2N}(T_1 + T_2)\left(1 + \frac{\log(2)}{\log(N)}\right).$$

Proof: Exercise. $\square$

4.8. Theorem (Dixmier traces):

Let $\omega : \ell^\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ be a linear map such that

(i) $\omega((\alpha_N)_{N=1}^\infty) \geq 0$ if $\alpha_N \geq 0$ for all $N \in \mathbb{N}$;

(ii) $\omega((\alpha_N)_{N=1}^\infty) = \lim_{N \to \infty} \alpha_N$ if $(\alpha_N)_{N=1}^\infty$ is convergent;

(iii) $\omega((\alpha_{2N})_{N=1}^\infty) = \omega((\alpha_N)_{N=1}^\infty)$ for each $(\alpha_N)_{N=1}^\infty \in \ell^\infty(\mathbb{N}, \mathbb{R})$ (scale invariance).

Put $\mathcal{L}^{(1,\infty)}(\mathcal{H})_+ := \{ T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) \mid T \geq 0 \}$ and define

$$\mathrm{Tr}_\omega(T) := \omega((\gamma_N(T))_{N=1}^\infty) \quad \text{for all } T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})_+.$$