① <u>Claim</u>: $d^*(f\omega) = f\,d^*\omega - df \llcorner \omega$

for $f \in \mathcal{A}$ and $\omega \in \Omega^\bullet_{\mathbb{C}}(M)$

<u>Proof</u>: We take $\eta \in \Omega^\bullet_{\mathbb{C}}(M)$ and compute with respect to the inner product $\langle \cdot,\cdot \rangle := \langle \cdot,\cdot \rangle_{\Omega^\bullet_{\mathbb{C}}(M)}$ that

$$\begin{aligned}
\langle d\eta, f\omega \rangle &= \langle \bar{f}\,d\eta, \omega \rangle \\
&= \langle d(\bar{f}\eta), \omega \rangle - \langle d\bar{f} \wedge \eta, \omega \rangle \\
&= \langle \bar{f}\eta, d^*\omega \rangle - \langle \eta, df \llcorner \omega \rangle \\
&= \langle \eta, f\,d^*\omega - df \llcorner \omega \rangle,
\end{aligned}$$

from which the assertion follows. $\square$

② <u>Claim</u>: $[D, \pi(f)]\omega = df \bullet \omega$

for $f \in \mathcal{A}$ and $\omega \in \Omega^\bullet_{\mathbb{C}}(M)$

<u>Proof</u>: We see that

$$\begin{aligned}
[D, \pi(f)]\omega &= [d, \pi(f)]\omega + [d^*, \pi(f)]\omega \\
&= \underbrace{d(f\omega) - f\,d\omega}_{= df \wedge \omega} + \underbrace{d^*(f\omega) - f\,d^*\omega}_{\overset{①}{=} -df \llcorner \omega} \\
&= df \wedge \omega - df \llcorner \omega \\
&= df \bullet \omega.
\end{aligned}$$

$\square$

That $[D, \pi(f)]$ extends to a bounded operator, will be discussed later.

Further, we note that $\Omega^\bullet_{\mathbb{C}}(M) \subseteq \operatorname{dom} d^*$, which justifies that $D_0 = d + d^*$ is densely defined with $\operatorname{dom} D_0 = \Omega^\bullet_{\mathbb{C}}(M)$.
This can be shown with the help of the <u>Hodge star operator</u> $* : \Omega^\bullet_{\mathbb{C}}(M) \to \Omega^\bullet_{\mathbb{C}}(M)$ which associates to each $\omega \in \Omega^k_{\mathbb{C}}(M)$ the unique $*\omega \in \Omega^{n-k}_{\mathbb{C}}(M)$ such that

$$(\overline{\omega} \wedge \eta)(x) = \langle \eta(x), (*\omega)(x) \rangle_{\Lambda^{n-k}_{\mathbb{C}} T^*_x M_{\mathbb{C}}} \, d\mathrm{vol}(x)$$

for all $\eta \in \Omega^{n-k}_{\mathbb{C}}(M)$ and all $x \in M$; in fact, one proves

$$d^*\big|_{\Omega^k_{\mathbb{C}}(M)} = (-1)^{nk+1} * d * .$$

That D_0 is essentially selfadjoint follows from results about general symmetric differential operators on manifolds (based on Friedrichs mollifiers).

In order to verify that D has compact resolvents, one defines for $s \geq 0$ the Sobolev spaces $H_s := \operatorname{dom} (1+\Delta)^{s/2}$, which are Hilbert spaces with respect to the inner product

$$\langle \omega, \eta \rangle_s := \langle \omega, \eta \rangle + \langle (1+\Delta)^{s/2}\omega, (1+\Delta)^{s/2}\eta \rangle$$

for $\omega, \eta \in H_s$ and uses the Rellich Lemma to show that $H_1 \hookrightarrow H_0$ is compact, which implies that

$$(1+\Delta)^{-1/2} : \mathcal{H} = H_0 \longrightarrow H_1 \hookrightarrow H_0 = \mathcal{H}$$

is compact and hence that $(D - i\mathbb{1})^{-1}$ is compact.

2.21. Remark:

If M carries more structure (i.e., if M is a spin^c manifold), then there is another spectral triple $(\mathcal{A}, \mathcal{H}, D)$ that is canonically associated to M, with D being the so-called Dirac operator.

We do not go into details here.

3. The geodesic distance in noncommutative geometry

Associating a spectral triple such as the Hodge-de Rham triple to a compact oriented smooth Riemannian manifold follows the philosophy of "spectral geometry", where such classical geometric objects are studied by spectral properties of canonically associated differential operators. At the same time, this allows to carry classical concepts over to a noncommutative setting. In this chapter, we discuss the geodesic distance within the framework of spectral triples.

3.1. Definition (geodesic distance):

Let (M, g) be a Riemannian manifold, i.e., M a paracompact smooth manifold with Riemannian metric g, and let $x_0, x_1 \in M$.

(i) We denote by $\Gamma(x_0, x_1)$ the set of all smooth paths $\gamma : [0,1] \to M$ satisfying $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Note that M is connected if and only if $\Gamma(x_0, x_1) \neq \emptyset$ for every choice of points $x_0, x_1 \in M$.

(ii) If $\gamma \in \Gamma(x_0, x_1)$ is given, we define $\gamma'(t) \in T_{\gamma(t)} M$ for each $t \in [0,1]$ by $\gamma'(t)([f]_{\gamma(t)}) := (f \circ \gamma)'(t)$ for all $[f]_{\gamma(t)} \in C^{\infty}_{\gamma(t)}(M)$; see Remark 2.3.

The <u>length</u> $L(\gamma)$ of γ is then defined as

$$L(\gamma) := \int_0^1 g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}\, dt.$$

(iii) Suppose that M is connected. The <u>geodesic distance</u> $d_g(x_0, x_1)$ between x_0 and x_1 is defined as

$$d_g(x_0, x_1) := \inf\{L(\gamma) \mid \gamma \in \Gamma(x_0, x_1)\}.$$

<u>3.2 Remark:</u>

(i) On a connected Riemannian manifold (M, g), the geodesic distance induces a metric

$$d_g : M \times M \to [0, \infty), \quad (x_0, x_1) \mapsto d_g(x_0, x_1),$$

called the <u>Riemannian distance function</u>. Note that $x_0 \neq x_1$ implies $d_g(x_0, x_1) > 0$. Indeed, for $\gamma \in \Gamma(x_0, x_1)$ and a local chart (U, φ) with $x_0 \in U$ and $x_1 \notin U$, we have

$$g_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \langle G(\gamma(t))\, v'(t), v'(t)\rangle \quad \forall t \in [0, T]$$

with $G = (g_{k,l})_{k,l=1}^n : \varphi(U) \to M_n(\mathbb{R})$ defined by

$$g_{k,l}(\varphi(x)) := g_x\big((d\varphi)(x)^{-1}(\partial_k|_{\varphi(x)}), (d\varphi)(x)^{-1}(\partial_l|_{\varphi(x)})\big)$$

for every $x \in U$ (see Theorem 2.17) and the smooth map

$$v : [0, T] \to \mathbb{R}^n, \quad t \mapsto \varphi(\gamma(t))$$

where $T \in (0,1]$ is chosen such that $\gamma([0,T]) \subset U$.

Take $r > 0$ such that $\overline{B_r(\varphi(x_0))} \subset \varphi(U)$ and put $V := \varphi^{-1}(B_r(\varphi(x_0)))$, which is an open subset of U.

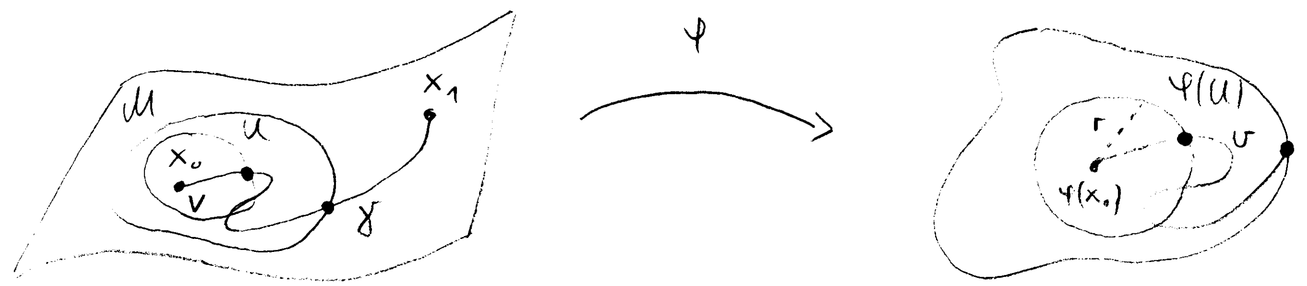
We find $\delta \in (0,1]$ such that

$$\delta |\xi| \leq \langle G(y)\xi, \xi\rangle^{1/2} \leq \delta^{-1} |\xi|$$

for all $y \in B_r(\varphi(x_0))$ and $\xi \in \mathbb{R}^n$. Thus

$$\begin{aligned} L(\gamma) &\geq \int_0^{T'} g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2} \, dt \\ &= \int_0^{T'} \langle G(\gamma(t))\, v'(t), v'(t)\rangle^{1/2} \, dt \\ &\geq \delta \int_0^{T'} |v'(t)| \, dt \\ &\geq \delta \left| \int_0^{T'} v'(t) \, dt \right| = \delta \, |v(T') - \varphi(x_0)| \end{aligned}$$

for every $T' \in (0,T]$ with $\gamma([0,T']) \subset V$.



By enlarging T and taking the limit in T', we infer $L(\gamma) \geq \delta r$ and thus $d_g(x_0, x_1) \geq \delta r > 0$.