(ii) The topology on M induced by the metric d_g agrees with the given topology on M.
This can be shown by arguments similar to (i).
In fact, one shows that for each $x_0 \in M$ and a local chart (U, φ) with $x_0 \in U$ an open neighborhood $V \subset U$ of x_0 and $\delta \in (0,1]$, $r > 0$ exist such that

$$\delta |\varphi(x) - \varphi(x_0)| \leq d_g(x, x_0) \leq \delta^{-1} |\varphi(x) - \varphi(x_0)|$$

for all $x \in V$ and $d_g(x, x_0) \geq \delta r$ for all $x \in M \setminus V$.

(iii) A connected paracompact smooth manifold is <u>second countable</u> (i.e. admits a countable base).
Thus, it follows from the Urysohn metrization theorem that the topology on M must be metrizable; this is in accordance with (ii).

(iv) Let (M_1, g_1) and (M_2, g_2) be connected Riemannian manifolds. Then, every isometry $\varphi : (M_1, d_{g_1}) \to (M_2, d_{g_2})$ (i.e., $d_{g_2}(\varphi(x_0), \varphi(x_1)) = d_{g_1}(x_0, x_1)$ for all $x_0, x_1 \in M_1$) is necessarily smooth and satisfies $\varphi^* g_2 = g_1$ (Myers-Steenrod theorem, 1939).
Note that if $\varphi : M_1 \to M_2$ is a smooth immersion between (paracompact) smooth manifolds M_1 and M_2 and if g is a Riemannian metric on M_2, then

$\varphi^* g$ is the Riemannian metric on $\mathcal{M}_1$ given by

$$(\varphi^* g)_x(\alpha,\beta) := g_{\varphi(x)}\big((d\varphi)(x)(\alpha),(d\varphi)(x)(\beta)\big)$$

for every $x \in \mathcal{M}_1$ and $\alpha,\beta \in T_x\mathcal{M}_1$, where the differential $(d\varphi)(x): T_x\mathcal{M}_1 \to T_{\varphi(x)}\mathcal{M}_2$ is defined as in Theorem 2.17; φ is called <u>immersion</u> if $(d\varphi)(x)$ is injective for each $x \in \mathcal{M}$.

Our goal is to "dualize" the definition of the geodesic distance such that it fits into the framework of spectral triples.

3.3. <u>Theorem</u> (musical isomorphisms)

Let $(\mathcal{M},g)$ be a Riemannian manifold. Then g can be seen as a positive definite pairing on smooth vector fields, i.e., a map

$$g: \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to C^\infty(\mathcal{M})$$

which is $C^\infty(\mathcal{M})$-bilinear and satisfies $g(X,X) \geq 0$ for all $X \in \mathfrak{X}(\mathcal{M})$ with $g(X,X)(x) = 0$ at $x \in \mathcal{M}$ if and only if $X(x) = 0$.

This induces an isomorphism (in fact, a $C^\infty(\mathcal{M})$-bimodule map)

$$\flat: \mathfrak{X}(\mathcal{M}) \to \Omega^1(\mathcal{M}),\quad X \mapsto X^\flat := g(X,\cdot).$$

Its inverse $\#: \Omega^1(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}), \omega \mapsto \omega^\#$ is determined by $\omega(X) = g(\omega^\#, X)$ for all $X \in \mathfrak{X}(\mathcal{M})$.

The inner product on $\Omega^1(\mathcal{M})$ defined in Remark 2.18 (iii) satisfies $\langle \omega,\eta\rangle_{\Omega^1(\mathcal{M})} = \int_{\mathcal{M}} g(\omega^\#,\eta^\#)$ for all $\omega,\eta \in \Omega^1(\mathcal{M})$, since $\langle \omega(x),\eta(x)\rangle_{T_x^*\mathcal{M}} = g_x(\omega^\#(x),\eta^\#(x))$ for each $x \in \mathcal{M}$.

3.4. Definition (gradient)

Let $(\mathcal{M}, g)$ be a Riemannian manifold. The gradient $\operatorname{grad} f$ of a function $f \in C^\infty(\mathcal{M})$ is the vector field

$$\operatorname{grad} f := (d^\circ f)^\# \in \mathfrak{X}(\mathcal{M})$$

with $d^\circ f \in \Omega^1(\mathcal{M})$ as defined in Definition 2.14 (iii). We thus have that for each $x \in \mathcal{M}$ and $\delta \in T_x\mathcal{M}$

$$g_x(\operatorname{grad}_x f, \delta) = (d^\circ f)(x)(\delta). \tag{3.1}$$

3.5. Definition:

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold. On $\mathfrak{X}(\mathcal{M})$, we define a norm $\|\cdot\|_\infty$ by

$$\|X\|_\infty := \max_{x \in \mathcal{M}} g_x(X(x), X(x))^{1/2} \quad \forall X \in \mathfrak{X}(\mathcal{M}).$$

3.6. Theorem:

Let $(\mathcal{M}, g)$ be a compact and connected Riemannian manifold. Then, for all $x_0, x_1 \in \mathcal{M}$, we have that

$$d_g(x_0, x_1) = \sup\left\{ |f(x_1) - f(x_0)| \;\middle|\; f \in C^\infty(\mathcal{M}) : \|\operatorname{grad} f\|_\infty \leq 1 \right\}$$

Proof:

Take any $\gamma \in \Gamma(x_0, x_1)$. Then, for every $f \in C^\infty(\mathcal{M})$,

$$f(x_1) - f(x_0) = f(\gamma(1)) - f(\gamma(0)) = \int_0^1 (f \circ \gamma)'(t)\, dt$$

and for each $t \in [0,1]$:

$$(f\circ\gamma)'(t) = (d^0 f)(\gamma(t))(\gamma'(t)) \overset{(3.1)}{=} g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \gamma'(t)),$$

so that the Cauchy-Schwarz inequality yields

$$|(f\circ\gamma)'(t)| \leq g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \operatorname{grad}_{\gamma(t)} f)^{1/2}\, g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}$$

$$\leq \|\operatorname{grad} f\|_\infty \cdot g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}.$$

In summary, we get

$$|f(x_1) - f(x_0)| \leq \|\operatorname{grad} f\|_\infty L(\gamma).$$

We infer from the latter that

$$\sup\{|f(x_1) - f(x_0)| \mid f \in C^\infty(M) : \|\operatorname{grad} f\|_\infty \leq 1\} \leq d_g(x_0, x_1).$$

In order to prove "$\geq$", we consider the function

$$f_0 : M \to \mathbb{R}, \quad x \mapsto d(x_0, x).$$

While f_0 is not (necessarily) smooth, the triangle inequality for d_g implies that f_0 is at least Lipschitz continuous with Lipschitz constant 1: For every $\varepsilon > 0$, we find $h_\varepsilon \in C^\infty(M)$ such that $\|f_0 - h_\varepsilon\|_\infty < \varepsilon$ and $\|\operatorname{grad} h_\varepsilon\|_\infty \leq 1 + \varepsilon$; put $f_\varepsilon := \frac{1}{1+\varepsilon} h_\varepsilon \in C^\infty(M)$. Then $\|\operatorname{grad} f_\varepsilon\|_\infty \leq 1$ and

$$|f_\varepsilon(x_1) - f_\varepsilon(x_0)| = \frac{1}{1+\varepsilon}\, |\underbrace{h_\varepsilon(x_1) - h_\varepsilon(x_0)}_{= f_0(x_1) - ((f_0(x_1) - h_\varepsilon(x_1)) + (h_\varepsilon(x_0) - \underbrace{f_0(x_0)}_{=0}))}|$$

$$\geq \frac{1}{1+\varepsilon}\Big(\underbrace{|f_0(x_n)|}_{=d_g(x_0,x_n)} - \big(\underbrace{|f_0(x_n)-h_\varepsilon(x_n)|}_{<\varepsilon} + \underbrace{|f_0(x_0)-h_\varepsilon(x_0)|}_{<\varepsilon}\big)\Big)$$

$$\geq \frac{1}{1+\varepsilon}\, d_g(x_0,x_n) - \frac{2\varepsilon}{1+\varepsilon}.$$

This shows that for every $\varepsilon > 0$

$$\sup\{|f(x_n)-f(x_0)| \mid f\in C^\infty(M) : \|\operatorname{grad} f\|_\infty \leq 1\} \geq \frac{d_g(x_0,x_n)}{1+\varepsilon} - \frac{2\varepsilon}{1+\varepsilon}.$$

We conclude by taking the limit $\varepsilon \downarrow 0$. □

This motivates the following definition:

3.7. Definition (Connes' spectral distance)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We define by

$$A := \overline{\pi(\mathcal{A})}^{\|\cdot\|} \subseteq B(\mathcal{H})$$

a C^*-algebra and denote by $S(A)$ the state space of A. We define for $\varphi, \psi \in S(A)$ by

$$d_D(\varphi,\psi) := \sup\{|\psi(\pi(a)) - \varphi(\pi(a))| \mid a\in\mathcal{A} : \|[D,\pi(a)]\|\leq 1\} \in [0,\infty]$$

the spectral distance between φ and ψ.

In view of Remark 3.2.(iv), the following theorem says that the Hodge-de Rham triple remembers the metric.

3.8. Theorem:

Let (M,g) be a compact and oriented Riemannian manifold and let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be the associated Hodge-de Rham triple as defined in Theorem 2.19. Then the faithful $*$-representation $\pi: \mathcal{A} \to B(\mathcal{H})$ extends to a faithful $*$-representation

$$\hat{\pi}: \; C(M,\mathbb{C}) \to B(\mathcal{H}),$$

which induces an isometric $*$-isomorphism

$$C(M,\mathbb{C}) \xrightarrow[\hat{\pi}]{\cong} A := \overline{\pi(\mathcal{A})}^{\|\cdot\|} \subseteq B(\mathcal{H}).$$

For $x \in M$, let $\delta_x \in S(A)$ be defined by $\delta_x(\hat{\pi}(f)) := f(x)$ for every $f \in C(M,\mathbb{C})$. If M is connected, then

$$d_g(x_0,x_1) = d_{\mathcal{D}}(\delta_{x_0},\delta_{x_1}) \quad \text{for all } x_0,x_1 \in M.$$

Proof:

① By definition of $\mathcal{H}$, it is easily seen that each $f \in C(M,\mathbb{C})$ defines an operator $\hat{\pi}(f) \in B(\mathcal{H})$ by $\hat{\pi}(f)\omega := f\omega$ for all $\omega \in \mathcal{H}$, with $\|\hat{\pi}(f)\| \leq \|f\|_\infty$; in fact, we have that $\|\hat{\pi}(f)\| = \|f\|_\infty$, since $\hat{\pi}(f)$ restricts to the ordinary multiplication operator on $L^2(M,g) := \overline{\Omega^0_{\mathbb{C}}(M)}$, for which we know $\|\hat{\pi}(f)|_{L^2(M,g)}\| = \|f\|_\infty$.

Thus, $\hat{\pi}: C(M,\mathbb{C}) \to B(\mathcal{H}),\ f \mapsto \hat{\pi}(f)$ is a faithful (isometric)

$*$-representation, which extends $\pi : \mathcal{A} \to B(\mathcal{H})$ and induces an isometric $*$-isomorphism $C(\mathcal{M}, \mathbb{C}) \cong A$.

(2) By equation (2.2) in Remark 2.20, we have for $f = \bar{f} \in \mathcal{A}$, $\omega \in \Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})$, and each $x \in \mathcal{M}$ that

$$\big([\mathcal{D}, \pi(f)]\,\omega\big)(x) = (df)(x) \cdot \omega(x) = c\big((df)(x)\big)\,\omega(x)$$

where, for each real $v \in T_x^* \mathcal{M}_{\mathbb{C}}$, the operator

$$c(v) : \Lambda_{\mathbb{C}}^{\bullet} T_x^* \mathcal{M}_{\mathbb{C}} \to \Lambda_{\mathbb{C}}^{\bullet} T_x^* \mathcal{M}, \quad w \mapsto v \cdot w$$

is an isometry; see Exercise 3B-1 (ii). Hence

$$\|([\mathcal{D}, \pi(f)]\,\omega)(x)\|_{\Lambda_{\mathbb{C}}^{\bullet} T_x^* \mathcal{M}_{\mathbb{C}}} \leq \|(df)(x)\|_{T_x^* \mathcal{M}_{\mathbb{C}}} \cdot \|\omega(x)\|_{\Lambda_{\mathbb{C}}^{\bullet} T_x^* \mathcal{M}_{\mathbb{C}}}.$$

We conclude that

$$\|[\mathcal{D}, \pi(f)]\,\omega\|_{\Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})} \leq \Big(\max_{x \in \mathcal{M}} \|(df)(x)\|_{T_x^* \mathcal{M}_{\mathbb{C}}}\Big) \|\omega\|_{\Omega_{\mathbb{C}}^{\bullet}(\mathcal{M})}$$

and so, by Definition 3.4 and Definition 3.5,

$$\|[\mathcal{D}, \pi(f)]\| \leq \max_{x \in \mathcal{M}} \|(df)(x)\|_{T_x^* \mathcal{M}_{\mathbb{C}}} = \|\operatorname{grad} f\|_\infty .$$

Optimizing ω, we get that $\|[\mathcal{D}, \pi(f)]\| = \|\operatorname{grad} f\|_\infty$ and thus, by Theorem 3.6 and Exercise 4AB-1 (ii),

$$\begin{aligned} d_g(x_0, x_1) &= \sup \{ |f(x_1) - f(x_0)| \mid f \in C^\infty(\mathcal{M}) : \|\operatorname{grad} f\|_\infty \leq 1 \} \\ &= \sup \{ |\delta_{x_1}(f) - \delta_{x_0}(f)| \mid f = \bar{f} \in \mathcal{A} : \|[\mathcal{D}, \pi(f)]\| \leq 1 \} \\ &= d_{\mathcal{D}}(\delta_{x_0}, \delta_{x_1}). \end{aligned}$$

$\square$