2.15. Definition

Let $U, V \subseteq \mathbb{R}^n$ be open. A diffeomorphism $f = (f_1, \ldots, f_n) : U \to V$ is said to be orientation preserving, if

$$\det \left( \left( \frac{\partial f_i}{\partial x_j}(x) \right)_{i,j=1}^n \right) > 0 \quad \forall x \in U.$$ 

2.16. Definition (orientation of smooth manifolds):

Let $M$ be a smooth manifold.

(i) A smooth atlas is called oriented, if all its transition maps are orientation preserving.

(ii) We say that $M$ is orientable, if it admits an oriented smooth atlas.

(iii) An orientation on $M$ is a maximal oriented smooth atlas.

2.17. Theorem (integration of smooth functions)

Let $M$ be an $n$-dimensional smooth manifold which is paracompact. Let $g$ be a Riemannian metric on $M$. Then there exists a unique linear map

$$\tilde{\mathcal{J}}_M : C_c^\infty(M) \to \mathbb{R}, \quad f \mapsto \int_M f$$
on the subring $C_c^\infty(M) \subseteq C^\infty(M)$ of compactly supported functions such that the following condition is satisfied:

For every local chart $(U, \varphi)$ in the maximal smooth atlas of $M$, the pullback $\varphi^* f$ is compactly supported in $U$ and

$$\tilde{\mathcal{J}}_M f = \int_U \varphi^* f.$$
of \( U \) and for each \( f \in C^\infty(U) \) with \( \text{supp}(f) \subseteq U \), we have that
\[
\int_U f = \int_{\Phi(U)} (\Phi \circ \Phi^{-1}) \sqrt{\det(g_{\Phi,\Phi})(x)} \, d\lambda^n \quad (2.1)
\]
where \( \lambda^n \) is the Lebesgue measure on \( \mathbb{R}^n \) and the functions \( g_{\Phi,\Phi} \in C^\infty(\Phi(U)) \) are determined by
\[
g_{\Phi,\Phi}(\Phi(x)) = \frac{1}{(d\Phi)(x)^{-1}} (\Theta(x \circ \Phi^{-1}(x)), (d\Phi)(x)^{-1}(d\lambda \circ \Phi(x)))
\]
for each \( x \in U \). Note that \( \{ \Theta_{\Phi,\Phi}(x) | \Phi = 1, \ldots, n \} \) is the basis of \( T_{\Phi(x)} \mathbb{R}^n \) introduced in Exercise 2(ii) on Sheet 1B and \( (d\Phi)(x) : T_x U \to T_{\Phi(x)} \mathbb{R}^n \) is the differential of \( \Phi \) at \( x \), which is defined by
\[
( (d\Phi)(x) \delta ) (\Phi \circ \Phi^{-1}) = \delta ( [\Phi \circ \Phi^{-1}]_x )
\]
for all \( \delta \in T_x U \) and \( [\Phi \circ \Phi^{-1}]_x \in C^\infty(\Phi(U)) \); in fact, \( (d\Phi)(x) \)

is bijective since \( \Phi \) is bijective.

2.18. Remark:

(i) In Exercise 1 on Sheet 3A, we will show that the right-hand side of (2.1) is well-defined, i.e., independent of the choice of the chart \((U, \Phi)\).

Using a partition of unity subordinate to \((U_i)_{i \in I}\)
for a maximal smooth atlas \( \mathcal{A} = \{(U_i, \Phi_i) | i \in I \} \), say
(i) i.e., one can thus define for general \( f \in C^\infty_c(M) \) \( \int
\):

\[
\int_M f = \sum_{i \in I} \int_{M_i} (p_i f),
\]

since \( \text{supp}(p_i f) \subset U_i \) for each \( i \in I \).

(ii) If \( M \) is orientable, Theorem 2.17 merges two different concepts:

- integration of compactly supported \( n \)-forms, i.e.

\[
\int_M : \Omega^n_c(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega,
\]

for an oriented paracompact \( n \)-dimensional smooth manifold \( M \).

- the volume form \( \text{d}v \in \Omega^n(M) \) of an oriented Riemannian manifold \((M, g)\) of dimension \( n \);

in general, \( \omega \in \Omega^k(M) \) is called a volume form if \( \omega \) vanishes nowhere, and a paracompact smooth manifold \( M \) is orientable if and only if a volume form exists; in fact, fixing an equivalence class of volume forms, specifies an orientation and vice versa; \( \text{d}v \) is chosen such that \( \text{d}v(x) \), for each \( x \in M \), is normalized with respect to the inner product \( \wedge^n T_x^* M \) induced by \( g \), i.e., \( \langle \text{d}v, \text{d}v \rangle_{\wedge^n T_x^* M} = 1 \).

One can show that \( \int_M f = \int_M \frac{\text{d}v}{\omega} f \) \( \forall f \in C^\infty_c(M) \).
(iii) Let $V$ be a finite dimensional real vector space and let $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ be an inner product on $V$. We then have an isomorphism

$$\bar{\Phi} : V \to V^*, \quad x \mapsto \langle \cdot , x \rangle$$

which allows us to define $\langle \cdot , \cdot \rangle_{V^*} : V^* \times V^* \to \mathbb{R}$ by

$$\langle \Phi, \psi \rangle_{V^*} := \langle \bar{\Phi}^{-1}(\Phi), \bar{\Phi}^{-1}(\psi) \rangle$$

for all $\Phi, \psi \in V^*$. For every $p \in \mathbb{N}$, we extend the latter to an inner product

$$\langle \cdot , \cdot \rangle_{\Lambda^p V^*} : \Lambda^p V^* \times \Lambda^p V^* \to \mathbb{R}$$

by

$$\langle \Phi_1 \wedge \ldots \wedge \Phi_p , \psi_1 \wedge \ldots \wedge \psi_p \rangle_{\Lambda^p V^*} := \det \left( \langle \Phi_b, \psi_e \rangle_{V^*} \right)_{b,e=1}^n$$

When applied to each fiber of $TM$ for an oriented paracompact smooth manifold $M$ with respect to the inner product induced by a Riemannian metric $g$ on $M$, we get for each $p > 0$ an inner product

$$\langle \cdot , \cdot \rangle_{\Omega^p_c(M)} : \Omega^p_c(M) \times \Omega^p_c(M) \to \mathbb{R}$$

(in the case $p = n$, $\langle \cdot , \cdot \rangle_{\Lambda^n T^*_x M}$ was used in (ii)) by

$$\langle \omega, \eta \rangle_{\Omega^p_c(M)} := \int_M \langle \omega(x), \eta(x) \rangle_{\Lambda^p T^*_x M} \, dvol(x)$$
for every $\omega, \eta \in \Omega^p_c(M)$. The latter extend naturally to inner products,

$$\langle \cdot, \cdot \rangle_{\Omega^p_c(M)} : \Omega^p_c(M) \times \Omega^p_c(M) \to \mathbb{C}.$$ 

2.19 Theorem (Hodge–de Rham triple):

Let $M$ be an oriented compact smooth manifold of dimension $n$ with Riemannian metric $g$. Consider

(i) the unital complex $*$-algebra $A := C^\infty(M, \mathbb{C})$;

(ii) the separable complex Hilbert space

$$H := L^2(\Lambda^\cdot_c T^* M, g),$$

which is obtained as the completion of the complex exterior algebra $\Omega^\cdot_c(M) := \bigoplus_{p=0}^n \Omega^p_c(M)$ with respect to the inner product given by

$$\langle (\omega_0, \ldots, \omega_n), (\eta_0, \ldots, \eta_n) \rangle_{\Omega^\cdot_c(M)} := \sum_{p=0}^n \langle \omega_p, \eta_p \rangle_{\Omega^p_c(M)}.$$ 

and the $*$-representation $\pi : A \to \mathcal{B}(H)$ given by multiplication, i.e., $\pi(f)\omega := f \omega$ for every $f \in A$ and $\omega \in H$;

(iii) the unbounded linear operator $D_0 := d + d^*$, where $d^*$ is the adjoint of the densely defined operator $d : H \supseteq dom d \to H$, $\omega \mapsto d\omega = d Re(\omega) + i d Im(\omega)$.
Then \( D \) is essentially self-adjoint; let \( \tilde{D} \) be its closure, which we call the **Hodge–de Rham operator**.

The **Hodge–de Rham triple** \((\Lambda, F, D)\) is a (commutative) spectral triple in the sense of Definition 1.1.

We call \( \Delta := \tilde{D}^2 \) the **Hodge Laplacian**.

2.20. **Remark:**

The proof of Theorem 2.19 relies mostly on techniques that are (not yet) at our disposal. We can understand, however, how commutators \([D, \pi(f)]\) for \( f \in \mathcal{A} \) look on \( \Lambda \); they are given by the **Clifford multiplication** with \( df \) from the left, i.e.,

\[
[D, \pi(f)] \omega = df \cdot \omega \quad \forall \omega \in \Omega^\bullet_C(M). \tag{2.2}
\]

The Clifford multiplication is defined on fibres as follows:

On \( \Lambda^p_C \): \( \bigoplus_{p \geq 0} \Lambda^p_C \) for the complexification \( V_C := V \otimes \mathbb{C} \) of a finite dimensional real Hilbert space \((V, \langle \cdot, \cdot \rangle)\), we define

\[
L : V_C \times \Lambda^p_C \to \Lambda^{p+1}_C
\]

by

\[
u \cdot (v_1 \wedge \ldots \wedge v_p) := \sum_{k=1}^p (-1)^{k+1} \langle v_k, \overline{\nu} \rangle v_1 \wedge \ldots \wedge \hat{v}_k \wedge \ldots \wedge v_p
\]

Then \( \nu \cdot \omega := \nu \wedge \omega - \nu L \omega \) for all \( \nu \in V_C \) and \( \omega \in \Lambda^p_C \).