2.15. Definition

Let $U, V \subseteq \mathbb{R}^n$ be open. A diffeomorphism $f=(f_1,\dots,f_n): U \to V$ is said to be orientation preserving, if

$$\det\left(\left(\partial_j f_i(x)\right)_{i,j=1}^{n} \right) > 0 \qquad \forall x \in U.$$

2.16. Definition (orientation of smooth manifolds):

Let M be a smooth manifold.

(i) A smooth atlas is called oriented, if all its transition maps are orientation preserving.

(ii) We say that M is orientable if it admits an oriented smooth atlas.

(iii) An orientation on M is a maximal oriented smooth atlas.

2.17. Theorem (integration of smooth functions)

Let M be an n-dimensional smooth manifold which is paracompact. Let g be a Riemannian metric on M. Then there exists a unique linear map

$$\int_M : C_c^\infty(M) \to \mathbb{R}, \quad f \mapsto \int_M f$$

on the subspace $C_c^\infty(M) \subseteq C^\infty(M)$ of compactly supported functions such that the following condition is satisfied:

For every local chart (U, φ) in the maximal smooth atlas of of $\mathcal{M}$ and for each $f \in C_c^\infty(\mathcal{M})$ with $\operatorname{supp}(f) \subset U$ we have that

$$\int_{\mathcal{M}} f = \int_{\varphi(U)} (f \circ \varphi^{-1}) \sqrt{\det(g_{k,l})_{k,l=1}^{n}} \, d\lambda^n \qquad (2.1)$$

Here, λ^n is the Lebesgue measure on $\mathbb{R}^n$ and the functions $g_{k,l} \in C^\infty(\varphi(U))$ are determined by

$$g_{k,l}(\varphi(x)) = g_x\Big((d\varphi)(x)^{-1}\big(\partial_k|_{\varphi(x)}\big), (d\varphi)(x)^{-1}\big(\partial_l|_{\varphi(x)}\big)\Big)$$

for each $x \in U$. Note that $\{\partial_k|_{\varphi(x)} \mid k = 1, \dots, n\}$ is the basis of $T_{\varphi(x)}\mathbb{R}^n$ introduced in Exercise 2(ii) on Sheet 1B and $(d\varphi)(x): T_x\mathcal{M} \to T_{\varphi(x)}\mathbb{R}^n$ is the <u>differential of φ at x</u>, which is defined by

$$\big((d\varphi)(x)\delta\big)\big([f]_{\varphi(x)}\big) = \delta\big([f \circ \varphi]_x\big)$$

for all $\delta \in T_x\mathcal{M}$ and $[f]_{\varphi(x)} \in C^\infty_{\varphi(x)}(\mathbb{R}^n)$; in fact, $(d\varphi)(x)$ is bijective since φ is bijective.

<u>2.18. Remark:</u>

(i) In Exercise 1 on Sheet 3A, we will show that the right-hand side of (2.1) is well-defined, i.e., independent of the choice of the chart (U, φ). Using a partition of unity subordinate to $(U_i)_{i \in I}$ for a maximal smooth atlas $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$, say

$(\rho_i)_{i\in I}$, one can thus define for general $f \in C_c^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} f = \sum_{i\in I} \int_{\mathcal{M}} (\rho_i f),$$

since $\operatorname{supp}(\rho_i f) \subset U_i$ for each $i \in I$.

(ii) If $\mathcal{M}$ is orientable, Theorem 2.17 merges two different concepts:

- integration of compactly supported n-forms, i.e.

$$\int_{\mathcal{M}} : \Omega_c^n(\mathcal{M}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathcal{M}} \omega,$$

for an oriented paracompact n-dimensional smooth manifold $\mathcal{M}$;

- the volume form $d\mathrm{vol} \in \Omega^n(\mathcal{M})$ of an oriented Riemannian manifold $(\mathcal{M}, g)$ of dimension n; in general, $\omega \in \Omega^n(\mathcal{M})$ is called <u>volume form</u> if ω vanishes nowhere, and a paracompact smooth manifold $\mathcal{M}$ is orientable if and only if a volume form exists; in fact, fixing an equivalence class of volume forms specifies an orientation and vice versa; $d\mathrm{vol}$ is chosen such that $d\mathrm{vol}(x)$, for each $x \in \mathcal{M}$, is normalized with respect to the inner product on $\Lambda^n T_x^*\mathcal{M}$ induced by g, i.e., $\langle d\mathrm{vol}, d\mathrm{vol}\rangle_{\Lambda^n T_x^*\mathcal{M}} = 1$.

One can show that $\int_{\mathcal{M}} f = \int_{\mathcal{M}} \underbrace{f\, d\mathrm{vol}}_{\in \Omega_c(\mathcal{M})} \quad \forall f \in C_c^\infty(\mathcal{M})$.

(iii) Let V be a finite dimensional real vector space and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be an inner product on V. We thus have an isomorphism

$$\Phi : V \to V^*, \quad x \mapsto \langle \bullet, x \rangle$$

which allows us to define $\langle \cdot, \cdot \rangle_{V^*} : V^* \times V^* \to \mathbb{R}$ by

$$\langle \varphi, \psi \rangle_{V^*} := \langle \Phi^{-1}(\varphi), \Phi^{-1}(\psi) \rangle$$

for all $\varphi, \psi \in V^*$. For every $p \in \mathbb{N}$, we extend the latter to an inner product

$$\langle \cdot, \cdot \rangle_{\Lambda^p V^*} : \Lambda^p V^* \times \Lambda^p V^* \to \mathbb{R}$$

by

$$\langle \varphi_1 \wedge \ldots \wedge \varphi_p, \psi_1 \wedge \ldots \wedge \psi_p \rangle_{\Lambda^p V^*} := \det \left(\langle \varphi_k, \psi_l \rangle_{V^*} \right)_{k,l=1}^{n}$$

When applied to each fibre of TM for an oriented paracompact smooth manifold M with respect to the inner product induced by a Riemannian metric g on M, we get for each $p \geq 0$ an inner product

$$\langle \cdot, \cdot \rangle_{\Omega^p_c(M)} : \Omega^p_c(M) \times \Omega^p_c(M) \to \mathbb{R}$$

(in the case $p = n$, $\langle \cdot, \cdot \rangle_{\Lambda^p T^*_x M}$ was used in (ii)) by

$$\langle \omega, \eta \rangle_{\Omega^p_c(M)} := \int_M \langle \omega(x), \eta(x) \rangle_{\Lambda^p T^*_x M} \, d\mathrm{vol}(x)$$

for every $\omega, \eta \in \Omega^n_c(\mathcal{M})$. The latter extend naturally to inner products

$$\langle \cdot, \cdot \rangle_{\Omega^p_{\mathbb{C},c}(\mathcal{M})} : \Omega^p_{\mathbb{C},c}(\mathcal{M}) \times \Omega^p_{\mathbb{C},c}(\mathcal{M}) \to \mathbb{C}.$$

<u>2.19 Theorem</u> (Hodge-de Rham triple):

Let $\mathcal{M}$ be an oriented compact smooth manifold of dimension n with Riemannian metric g. Consider

(i) the unital complex $*$-algebra $\mathcal{A} = C^\infty(\mathcal{M}, \mathbb{C})$;

(ii) the separable complex Hilbert space

$$\mathcal{H} := L^2(\Lambda^\bullet_{\mathbb{C}} T^* \mathcal{M}, g),$$

which is obtained as the completion of the complex exterior algebra $\Omega^\bullet_{\mathbb{C}}(\mathcal{M}) := \bigoplus_{p=0}^n \Omega^p_{\mathbb{C}}(\mathcal{M})$ with respect to the inner product given by

$$\langle (\omega_0, \dots, \omega_n), (\eta_0, \dots, \eta_n) \rangle_{\Omega^\bullet_{\mathbb{C}}(\mathcal{M})} := \sum_{p=0}^n \langle \omega_p, \eta_p \rangle_{\Omega^p_{\mathbb{C}}(\mathcal{M})},$$

and the $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ given by multiplication, i.e., $\pi(f)\omega := f\omega$ for every $f \in \mathcal{A}$ and $\omega \in \mathcal{H}$;

(iii) the unbounded linear operator $D_0 := d + d^*$, where d^* is the adjoint of the densely defined operator

$$d : \mathcal{H} \supseteq \operatorname{dom} d \to \mathcal{H}, \quad \omega \mapsto d\omega = d\,\mathrm{Re}(\omega) + i\, d\,\mathrm{Im}(\omega)$$

with domain $\mathrm{dom}\, d := \Omega^{\bullet}_{\mathbb{C}}(M)$.

Then D_0 is essentially selfadjoint; let D be its closure, which we call the Hodge-de Rham operator.
The Hodge-de Rham triple $(\mathcal{A}, \mathcal{H}, D)$ is a (commutative) spectral triple in the sense of Definition 1.1.
We call $\Delta := D^2$ the Hodge Laplacian.

2.20. Remark:

The proof of Theorem 2.19 relies mostly on techniques that are (not yet) at our disposal. We can understand, however, how commutators $[D, \pi(f)]$ for $f \in \mathcal{A}$ look on $\mathcal{H}$; they are given by the Clifford multiplication with df from the left, i.e.,

$$[D, \pi(f)]\,\omega = df \cdot \omega \qquad \forall\, \omega \in \Omega^{\bullet}_{\mathbb{C}}(M). \quad (2.2)$$

The Clifford multiplication is defined on fibres as follows: on $\Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}} := \bigoplus_{p \geq 0} \Lambda^{p}_{\mathbb{C}} V_{\mathbb{C}}$ for the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ of a finite dimensional real Hilbert space $(V, \langle \cdot, \cdot \rangle)$, we define

$$L : V_{\mathbb{C}} \times \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}} \to \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}$$

$v = u \otimes \lambda \in V_{\mathbb{C}}$: $\bar{v} := u \otimes \bar{\lambda}$

by $$v L (v_1 \wedge \ldots \wedge v_p) := \sum_{k=1}^{p} (-1)^{k+1} \langle v_k, \bar{v} \rangle_{\mathbb{C}}\, v_1 \wedge \ldots \wedge \hat{v}_k \wedge \ldots \wedge v_p$$

$\langle u_1 \otimes \lambda_1, u_2 \otimes \lambda_2 \rangle_{\mathbb{C}} := \langle u_1, u_2 \rangle \lambda_1 \bar{\lambda}_2$

Then $v \cdot w := v \wedge w - v L w$ for all $v \in V_{\mathbb{C}}$ and $w \in \Lambda^{\bullet}_{\mathbb{C}} V_{\mathbb{C}}$.