UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 1
Hand in on Friday, 26.04.19, before the lecture.

In Examples 1.7 and 1.8 in class we saw two realizations of the most important nc distribution, namely n free semicircular elements. In this assignment you are asked to familiarize yourself with the meaning of this. For the notion of freeness you might watch Lecture 1 and 2 from the class "Free Probability Theory" from last term or read the corresponding Section 1 of the class notes. For random matrices you might watch Lecture 17 and 18 or read Section 6.

Exercise 1 (10 points)**.**
Let $S_1, \dots, S_n$ be the operators on the full Fock space from Example 1.7.

i) Saying that each $S \in \{S_i \colon 1 \leq i \leq n\}$ is a semicircular variable means that its odd moments are zero and the even moments are given by the Catalan numbers, i.e.

$$\varphi(S^{2k+1}) = 0 \quad \text{and} \quad \varphi(S^{2k}) = \frac{1}{k+1}\binom{2k}{k}.$$

Check the latter for small k, i.e. show that

$$\varphi(S^2) = 1, \qquad \varphi(S^4) = 2, \qquad \varphi(S^6) = 5, \qquad \varphi(S^8) = 14.$$

ii) Saying that the $S_1, \dots, S_n$ are free means that special mixed moments vanish. Show this for the following special cases.

$$\varphi(S_1 S_2 S_1 S_2) = 0, \qquad \varphi((S_1^4 - 2)(S_2^6 - 5)(S_1^2 - 1)) = 0.$$

Exercise 2 (10 points)**.**
Let $X_i^{(N)}$ be the independent Gaussian random matrices from Example 1.8. Familiarize yourself with computer programs (e.g., matlab) to produce random matrices and calculate and plot histograms of their eigenvalues.

i) Saying that, for each i, $X_i^{(N)}$ is asymptotically a semicircular variable means that for large N the eigenvalue distribution of the N eigenvalues of such a matrix is close to the semicircle distribution. Check this by producing a histogram for a 1000×1000 Gaussian random matrix.

ii) Saying that $X_1^{(N)}, \dots, X_n^{(N)}$ are asymptotically free means that special mixed moments (with respect to the normalized trace tr) are, for large N, close to zero. Check this numerically for the following special cases:

$$\operatorname{tr}(ABAB), \qquad \varphi((A^4 - 2)(B^6 - 5)(A^2 - 1)),$$

where A and B are two independent 1000×1000 Gaussian random matrices.