UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 2
Hand in on Friday, 03.05.19, before the lecture.

Exercise 1 (20 points).
Let $(\mathcal{C}, \varphi)$ be a non-commutative probability space. Put

$$\mathcal{A} := M_n(\mathcal{C}), \qquad \mathcal{B} := M_n(\mathbb{C})$$

and

$$E := \mathrm{id} \otimes \varphi : \mathcal{A} \to \mathcal{B}.$$

i) Show that $(\mathcal{A}, \mathcal{B}, E)$ is an operator-valued probability space.

ii) Assume that $(\mathcal{C}, \varphi)$ is a C^*-probability space. Show that $(\mathcal{A}, \mathcal{B}, E)$ is then an operator-valued C^*-probability space.

iii) Show that in the C^*-case we also have: if φ is faithful, then E is also faithful. [Faithful means: $E(A^*A) = 0$ implies that $A = 0$.]

iv) Assume that φ is a trace, i.e., $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathcal{C}$. Does then also E have the tracial property? Give a proof or counter example!

Exercise 2 (20 points).
Let $\mathcal{B}$ be a unital algebra. Consider a collection of functions $F = (F_m)_{m \in \mathbb{N}}$

$$F_m : M_m(\mathcal{B}) \to M_m(\mathcal{B}), \quad z \mapsto F_m(z).$$

i) We say that F respects direct sums if

$$F_{m_1+m_2} \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} F_{m_1}(z_1) & 0 \\ 0 & F_{m_2}(z_2) \end{pmatrix}$$

for all $m_1, m_2 \in \mathbb{N}$, $z_1 \in M_{m_1}(\mathcal{B})$, $z_2 \in M_{m_2}(\mathcal{B})$.

ii) We say that F respects similarities if

$$F_m(SzS^{-1}) = SF_m(z)S^{-1}$$

for all $m \in \mathbb{N}$ and all $S \in M_m(\mathbb{C})$ invertible.

iii) We say that F respects intertwininigs if for all $n, m \in \mathbb{N}$, $z_1 \in M_n(\mathcal{B})$, $z_2 \in M_m(\mathcal{B})$, $T \in M_{n,m}(\mathbb{C})$ (the latter are the $n \times m$ matrices with complex entries) we have the following:

$$z_1 T = T z_2 \implies F_n(z_1)T = T F_m(z_2).$$

Prove that [(i) and (ii)] is equivalent to (iii).