UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 3
Hand in on Friday, 17.05.19, before the lecture.

Exercise 1 (10 points).
Prove the second item from the proof of Lemma 3.6: Let f be a non-commutative function, then we have for $z_1 \in M_n(\mathcal{B})$, $z_2 \in M_m(\mathcal{B})$ that

$$\partial f(z_1, z_2)\sharp(w_1 + w_2) = \partial f(z_1, z_2)\sharp w_1 + \partial f(z_1, z_2)\sharp w_2$$

for all $w_1, w_2 \in M_{n,m}(\mathcal{B})$.

Exercise 2 (10 points).
Let $r \in \mathbb{N}$ and $b_0, b_1, \ldots, b_{r+1} \in \mathcal{B}$ be given and consider the monomial f

$$f(z) = b_0 z b_1 z b_2 z \cdots b_r z b_{r+1}.$$

(i) Show that $f = (f_m)_{m\in\mathbb{N}}$ is a non-commutative function. (For this, also give first the precise definition of all $f_m : M_m(\mathcal{B}) \to M_m(\mathcal{B})$.)
(ii) Calculate the first and second order derivatives of f, i.e.,

$$\partial f(z_1, z_2)\sharp w, \qquad \text{and} \qquad \partial^2 f(z_1, z_2, z_3)\sharp(w_1, w_2).$$

Exercise 3 (10 points).
For a non-commutative function f we define the mappings $\partial^{k-1}(z_1, \ldots, z_k)\sharp(w_1, \ldots, w_{k-1})$ by

$$f\left(\begin{pmatrix} z_1 & w_1 & 0 & \ldots & 0 \\ 0 & z_2 & w_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & z_{k-1} & w_{k-1} \\ 0 & 0 & \ldots & 0 & z_k \end{pmatrix}\right)$$

$$= \begin{pmatrix} f(z_1) & \partial f(z_1, z_2)\sharp w_1 & \partial^2(z_1, z_2, z_3)\sharp(w_1, w_2) & \ldots & \partial^{k-1} f(z_1, \ldots, z_k)\sharp(w_1, \ldots, w_{k-1}) \\ 0 & f(z_2) & \partial f(z_2, z_3)\sharp w_2 & \ldots & \partial^{k-2} f(z_2, \ldots, z_k)\sharp(w_2, \ldots, w_{k-1}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \partial f(z_{k-1}, z_k)\sharp w_{k-1} \\ 0 & 0 & 0 & \ldots & f(z_k) \end{pmatrix}$$

Show that for each $N \in \mathbb{N}$ we have the expansion

$$f(z+tw) = \sum_{k=0}^{N} t^k \partial^k(z,\dots,z,z)\sharp(w,\dots,w) + t^{N+1}\partial^{N+1} f(z,\dots,z,z+tw)\sharp(w,\dots,w)$$

for $m \in \mathbb{N}$, $z, w \in M_m(\mathcal{B})$ and $t \in \mathbb{C}$. (You can assume for this that $\partial^{k-1}(z_1,\dots,z_k)\sharp(w_1,\dots,w_{k-1})$ is linear in the arguments w_i.)
Hint: It might be helpful, to consider the matrix

$$y := \begin{pmatrix} z & tw & 0 & \dots & 0 & 0 \\ 0 & z & tw & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z & tw \\ 0 & 0 & 0 & \dots & 0 & z+tw \end{pmatrix}$$

and observe that

$$y \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \big(z+tw\big)$$

Exercise 4 (10 points)**.**
Consider the C^*-algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$. We define its upper half-plane by

$$\mathbb{H}^+(M_n(\mathbb{C})) := \{b \in M_n(\mathbb{C}) \mid \exists \varepsilon > 0 : \operatorname{Im}(b) \geq \varepsilon 1\},$$

where $\operatorname{Im}(b) := (b - b^*)/(2i)$.
(i) In the case $n = 2$, show that in fact

$$\mathbb{H}^+(M_2(\mathbb{C})) := \left\{ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \middle| \operatorname{Im}(b_{11}) > 0, \operatorname{Im}(b_{11}) \operatorname{Im}(b_{22}) > \frac{1}{4}|b_{12} - \overline{b_{21}}|^2 \right\}.$$

(ii) For general $n \in \mathbb{N}$, prove: if a matrix $b \in M_n(\mathbb{C})$ belongs to $\mathbb{H}^+(M_n(\mathbb{C}))$ then all eigenvalues of b lie in the complex upper half-plane $\mathbb{C}^+$. Is the converse also true?