UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 4
Hand in on Friday, 24.05.19, before the lecture.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *completely positive* if all matrix amplifications $\Phi \otimes \mathrm{id} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ are positive.

Exercise 1 (10 points)**.**
Show that the following are equivalent:

i) $\Phi : \mathcal{A} \to \mathcal{B}$ is completely positive.

ii) For each $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in \mathcal{A}$ the matrix $(\Phi(a_i a_j^*))_{i,j=1}^n \in M_n(\mathcal{B})$ is positive.

Exercise 2 (10 points)**.**
Show that the transpose map on 2×2 matrices,

$$\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix},$$

is positive, but not completely positive.

Exercise 3 (10 points)**.**
Show that a positive conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is completely positive. [Hint: For this you can use the following statement: A matrix $(b_{ij})_{i,j=1}^n \in M_n(\mathcal{B})$ is positive if and only if we have

$$\sum_{i,j=1}^n b_i b_{ij} b_j^* \geq 0 \qquad \text{for all } b_1, \dots, b_n \in \mathcal{B}.]$$

What does this tell us about the complete positivity of states $\varphi : \mathcal{A} \to \mathbb{C}$?

Exercise 4 (10 points)**.**

i) Let $(\mathcal{A}, \mathcal{B}, E)$ be a $\mathcal{B}$-valued C^*-probability space. Consider a "constant" selfadjoint random variable $b = b^* \in \mathcal{B} \subset \mathcal{A}$. Calculate the fully matricial Cauchy transform of b.

ii) Consider a C^*-probability space $(\mathcal{A}, \varphi)$ as a special case of an operator-valued C^*-probability space, where $\mathcal{B} = \mathbb{C}$. Consider a selfadjoint $X = X^* \in \mathcal{A}$. Its distribution μ_X is then a probability measure on $\mathbb{R}$. Express the fully matricial $\mathbb{C}$-valued Cauchy transform G_X in terms of μ_X.