UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 6
Hand in on Friday, June 14th using mailbox no. 114, building E2 5.

Exercise 1 (20 points)**.**
Let $\eta : \mathcal{B} \to \mathcal{B}$ be a completely positive map on the C^*-algebra $\mathcal{B}$. We want to construct an operator X which has η as its second moment; this will be a kind of operator-valued Bernoulli element. For this we consider the degenerate Fock space

$$\mathcal{F} := \mathcal{B} \oplus \mathcal{B}x\mathcal{B} \subset \mathcal{B}\langle x\rangle,$$

equipped with the $\mathcal{B}$-valued inner product

$$\langle \quad , \quad \rangle : \mathcal{F} \times \mathcal{F} \to \mathcal{B},$$

given by linear extension of

$$\langle b_0 + b_1 x b_2, \tilde{b}_0 + \tilde{b}_1 x \tilde{b}_2 \rangle := b_0^* \tilde{b}_0 + b_2^* \eta(b_1^* \tilde{b}_1) \tilde{b}_2.$$

On $\mathcal{F}$ we define the creation operator l^* by

$$l^* b = xb \qquad l^* b_1 x b_2 = 0,$$

and the annihilation operator l by

$$lb = 0, \qquad l b_1 x b_2 = \eta(b_1) b_2.$$

Let $\mathcal{A}$ be the $*$-algebra which is generated by l and by elements $b \in \mathcal{B}$ acting as left multiplication operators on $\mathcal{F}$. We also put

$$E : \mathcal{A} \to \mathcal{B}, \qquad A \mapsto E[A] := \langle 1, A1 \rangle.$$

i) Show that the inner product is positive and that l and l^* are adjoints of each other.
ii) Show that E is positive.
iii) Show that the second moment of the selfadjoint operator $X = l + l^*$ is given by η.
iv) What is the formula for a general moment of X.

Exercise 2 (10 points)**.**
Let $S \in \mathcal{A}$ be a $\mathcal{B}$-valued semicircular element with covariance $\eta : \mathcal{B} \to \mathcal{B}$. Fix $n \in \mathbb{N}$ and $b \in M_n(\mathcal{B})$. Consider now

$$\hat{S} := b(1 \otimes S)b^* = b \begin{pmatrix} S & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & S \end{pmatrix} b^* \in M_n(\mathcal{A}).$$

Show that $\hat{S}$ is an $M_n(\mathcal{B})$-valued semicircular element and calculate its covariance

$$\hat{\eta} : M_n(\mathcal{B}) \to M_n(\mathcal{B}).$$

Exercise 3 (10 points)**.**
Assume that we have $X_i^{(1)}$ ($i \in \mathbb{N}$) which are f.i.d., with first moment zero and second moment given by a covariance $\eta_1 : \mathcal{B} \to \mathcal{B}$; and that we have $X_i^{(2)}$ ($i \in \mathbb{N}$) which are f.i.d with first moment zero and second moment given by a covariance $\eta_2 : \mathcal{B} \to \mathcal{B}$. According to the operator-valued version of the free central limit theorem we know then that the normalized sum of the $X_i^{(1)}$ converges to an operator-valued semicircular element S_1 with covariance η_1 and that the normalized sum of the $X_i^{(2)}$ converges to an operator-valued semicircular element S_2 with covariance η_2.
Assume now that the $X_i^{(1)}$ and $X_i^{(2)}$ are realized in the same C^*-probability space and are also free for each i. Then the joint distribution of $(X_i^{(1)}, X_i^{(2)})$ converges to the joint distribution of the pair (S_1, S_2). Convince yourself that our argument (from last term for the scalar-valued case) that freeness goes over to the limit remains valid in the operator-valued case. Thus we get in the limit two semicircular elements which are free.
By repeating the calculation in our proof of the central limit theorem for this multivariate setting derive the formula for mixed moments of two free semicircular elements S_1 and S_2, with covariance mappings η_1 and η_2, respectively.