UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 7
Hand in on Friday, June 21st using Mailbox no. 14, building E2 5.

Exercise 1 (20 points).
Let $\eta : \mathcal{B} \to \mathcal{B}$ be a completely positive map on the C^*-algebra $\mathcal{B}$. We want to construct a semicircular operator X which has ν_η as its distribution. This operator will be constructed on an operator-valued version of the full Fock space. The latter is nothing but our polynomials $\mathcal{B}\langle x\rangle$, equipped with the $\mathcal{B}$-valued inner product

$$\langle b_0xb_1x\cdots b_nxb_{n+1}, \tilde{b}_0x\tilde{b}_1x\cdots\tilde{b}_mx\tilde{b}_{m+1}\rangle := \delta_{nm}b_{n+1}^*\eta\left(b_n^*\cdots\eta\big(b_1^*\eta(b_0^*\tilde{b}_0)\tilde{b}_1\big)\cdots\tilde{b}_n\right)\tilde{b}_{n+1}.$$

On this full fock space $\mathcal{F}$ we define again a creation operator l^*, now given by

$$l^*b_0xb_1\cdots xb_{n+1} := xb_0xb_1\cdots xb_{n+1},$$

and an annihilation operator l^*, given by $lb = 0$ and

$$lb_0xb_1x\cdots xb_{n+1} := (\eta(b_0)b_1)x\cdots xb_{n+1}.$$

Elements from $\mathcal{B}$ act on $\mathcal{F}$ by left multiplication. For $\mathcal{A}$ we take now the $*$-algebra which is generated by l and by all multiplication operators from $\mathcal{B}$. Furthermore, we put

$$E : \mathcal{A} \to \mathcal{B}, \qquad A \mapsto E[A] := \langle 1, A1\rangle.$$

i) Show that the inner product on $\mathcal{F}$ is positive and that l and l^* are adjoints of each other.
ii) Show that E is positive.
iii) Calculate explicitly the second and the fourth moments of $X := l + l^*$.
iv) Prove that $X = l + l^*$ has semicircular distribution ν_η.

Exercise 2 (20 points).
Let S_1 and S_2 be two free (scalar-valued) standard semicircular elements and consider

$$S := \begin{pmatrix} 0 & S_1 \\ S_1 & S_2 \end{pmatrix}.$$

We have seen in class (Part (3) of Remark 6.3) that S is then an $M_2(\mathbb{C})$-valued semicircular element whose covariance function $\eta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is given by

$$\eta\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} + b_{22} \end{pmatrix}$$

Refresh your memory on the relation between free semicircular elements and independent GUE random matrices (for example, from Section 6 of the Free Probability class). From this it follows that S is the limit of a random matrix

$$X_N = \begin{pmatrix} 0 & A_N \\ A_N & B_N \end{pmatrix},$$

where A_N and B_N are independent GUE random matrices. (If A_N and B_N are $N \times N$ matrices, then X_N is of course a $2N \times 2N$ matrix.) Since

$$g(z) = \operatorname{tr} E[(z - S)^{-1}] = \operatorname{tr} G(z)$$

is the scalar-valued Cauchy transform of S with respect to $\operatorname{tr} \circ E$ (tr is here the normalized trace over 2×2 matrices), we can calculate the Cauchy transform $g(z)$ of the limiting eigenvalue distribution of X_N by first calculating the $M_2(\mathbb{C})$-valued Cauchy transform $G(z)$ of S and then taking the trace of this. For invoking the Cauchy-Stieltjes inversion formula, we should calculate this for z close to the real axis.

i) We know that the operator-valued Cauchy transform (on the ground level) $G(b)$ satisfies the matrix equation

$$bG(b) = 1 + \eta(G(b))G(b).$$

This is true for all $b \in M_2(\mathbb{C})$, but we are here only interested in arguments of the form $b = z1$, where $z \in H^+(\mathbb{C})$. Try to solve this equation (exactly or numerically) for $z \in H^+(\mathbb{C})$ close to the real axis, so that you can produce from this a density for the scalar-valued distribution of S.

ii) Realize for large N the random matrix X_N and calculate histograms for its eigenvalue distribution. Compare this with the result from (i).