UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 8
Hand in on Friday, July 5th, 2019, before the lecture.

Exercise 1 (10 points).
Prove Proposition 7.2 from class: Let $(\mathcal{A}, \varphi)$ be a C^*-probability space and $S_1, \dots, S_d \in \mathcal{A}$ free standard semicirculars (i.e., $\varphi(S_i^2) = 1$). For $n \geq 1$ and selfadjoint matrices $b_1, \dots, b_d \in M_n(\mathbb{C})$ we consider

$$S := b_1 \otimes S_1 + \cdots + b_d \otimes S_d \in M_n(\mathcal{A}).$$

Then S is in the matrix-valued C^*-probability space $(M_n(\mathcal{A}), M_n(\mathbb{C}), \mathrm{id} \otimes \varphi)$ a matrix-valued semicircular element with covariance $\eta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$\eta(b) = \sum_{j=1}^{d} b_j b b_j.$$

Exercise 2 (10 points).
Let S_{ij} for $i \geq j$ be free standard semicircular elements, and put $S_{ij} = S_{ji}$. Furthermore, let $\alpha_{ij} \in \mathbb{R}$ with $\alpha_{ij} = \alpha_{ji}$ be given. Then we consider

$$S := (\alpha_{ij} S_{ij})_{i,j=1}^{n}.$$

From the previous exercise we know that this is an $M_n(\mathbb{C})$-valued semicircular element. Give, by relying on Theorem 7.4, a criterium to decide whether S is also a scalar-valued semicircular element. Use this to decide whether the following are scalar-valued semicircular elements (for $S_1, \dots, S_6$ free standard semicirculars):

$$S = \begin{pmatrix} 3S_1 & 0 & 4S_2 \\ 0 & 5S_3 & 0 \\ 4S_2 & 0 & 3S_4 \end{pmatrix} \qquad \text{or} \qquad \tilde{S} = \begin{pmatrix} 3S_1 & 6S_5 & 4S_2 \\ 6S_5 & 5S_3 & 6S_6 \\ 4S_2 & 6S_6 & 3S_4 \end{pmatrix}.$$

Exercise 3 (10 points).
Check your conclusion from the last exercise numerically by producing histograms, for $N = 1000$ or higher, of the eigenvalues of the matrices

$$X^{(3N)} = \begin{pmatrix} 3X_1^{(N)} & 0 & 4X_2^{(N)} \\ 0 & 5X_3^{(N)} & 0 \\ 4X_2^{(N)} & 0 & 3X_4^{(N)} \end{pmatrix} \qquad \text{or} \qquad \tilde{X}^{(3N)} = \begin{pmatrix} 3X_1^{(N)} & 6X_5^{(N)} & 4X_2^{(N)} \\ 6X_5^{(N)} & 5X_3^{(N)} & 6X_6^{(N)} \\ 4X_2^{(N)} & 6X_6^{(N)} & 3X_4^{(N)} \end{pmatrix},$$

where $X_1^{(N)}, \dots, X_6^{(N)}$ are independent $GUE(N)$ random matrices.