UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture on
Non-Commutative Distributions
Summer Term 2019

Assignment 9
Hand in on Friday, July 12th, 2019, before the lecture.

Exercise 1 (10 points).
Write down the precise form of the recursion between moments and free cumulants from Proposition 9.7, and prove this by checking that the arguments from the scalar-valued case work also in the operator-valued situation.

Exercise 2 (10 points).
Write down explicitly the linearization for a monomial of degree $k = 5$, as given in the proof of Theorem 8.5 and check that this satisfies indeed all the requirements for a linearization.

Exercise 3 (10 points).
Find a linearization $\hat{p}$ of the polynomial

$$p(x, y) = xy^2 + y^2x - y.$$

Bonus Questions:

Exercise* 4 (20 bonus points).
Calculate, via linearization and numerical calculation of the corresponding operator-valued semicircular or of the corresponding operator-valued free convolution, the distribution of $p(X, Y) = XY^2 + Y^2X - Y$, where

- X and Y are free standard semicircular elements
- X and Y are free random variables, with

$$\mu_X = \frac{1}{2}(\delta_0 + \delta_1), \qquad \mu_Y = \frac{1}{2}(\delta_{-1} + \delta_1).$$

Exercise* 5 (10 bonus points).
Realize X and Y, as given in Exercise 4, (asymptotically) via large $N \times N$ random matrices X_N and Y_N, and produce histograms of the eigenvalue distribution of $p(X_N, Y_N)$. Compare the results with the calculations from Exercise 4.