Assignment 1 A

Exercise 1

(i) $$\left(\frac{\partial}{\partial x_i} f \circ \phi\right)(x) = \sum_{j=1}^{N} \frac{\partial f}{\partial y_j}(\phi(x)) \underbrace{\frac{\partial \phi_j}{\partial x_i}(x)}_{= Q_{ji}},$$

since $$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{pmatrix} = a + Qx = \begin{pmatrix} a_1 + \sum_{j=1}^{N} Q_{1j} x_j \\ \vdots \\ a_N + \sum_{j=1}^{N} Q_{Nj} x_j \end{pmatrix}$$

$$\left(\frac{\partial^2}{\partial x_i^2} f \circ \phi\right)(x) = \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} Q_{ji} \frac{\partial f}{\partial y_j}(\phi(x))\right)$$

$$= \sum_{j=1}^{N} Q_{ji} \sum_{r=1}^{N} \frac{\partial^2 f}{\partial y_r \partial y_j}(\phi(x)) \underbrace{\frac{\partial \phi_r}{\partial x_i}(x)}_{= Q_{ri}}$$

$$= \sum_{j,r=1}^{N} \frac{\partial^2 f}{\partial y_r \partial y_j}(\phi(x))\, Q_{ji} Q_{ri}$$

$$\Longrightarrow \quad (\Delta(f \circ \phi))(x) = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} f \circ \phi \right)(x)$$

$$= \sum_{j,r=1}^{N} \frac{\partial^2 f}{\partial y_r \partial y_j}(\phi(x)) \underbrace{\left(\sum_{i=1}^{N} Q_{ji} Q_{ri} \right)}_{= \delta_{j,r}}$$

$$= (\Delta f)(\phi(x))$$

So, if $f \in H(\phi(\Omega))$, then $f \circ \phi \in H(\Omega)$.

(ii) $$\left(\frac{\partial}{\partial x_i} f \circ \phi\right)(x) = \sum_{j=1}^{N} \frac{\partial f}{\partial y_j}(\phi(x)) \underbrace{\frac{\partial \phi_j}{\partial x_i}(x)}_{= \delta_{ij}\alpha} = \alpha \frac{\partial f}{\partial y_i}(\phi(x))$$

$$\left(\frac{\partial^2}{\partial x_i^2} f \circ \phi\right)(x) = \frac{\partial}{\partial x_i}\left(\alpha \frac{\partial f}{\partial y_i}(\phi(x))\right)$$

$$= \alpha \sum_{j=1}^{N} \frac{\partial^2 f}{\partial y_j \partial y_i}(\phi(x)) \underbrace{\frac{\partial \phi_j}{\partial x_i}(x)}_{= \delta_{ij}\alpha} = \alpha^2 \frac{\partial^2 f}{\partial y_i^2}(\phi(x))$$

$$\Longrightarrow (\Delta(f \circ \phi))(x) = \alpha^2 (\Delta f)(\phi(x))$$

So, if $f \in H(\phi(\Omega))$, then $f \circ \phi \in H(\Omega)$.

Exercise 2

Let $f, g \in C^2(\Omega)$ be given. Then

$$\frac{\partial}{\partial x_i}(f \cdot g)(x) = \frac{\partial f}{\partial x_i}(x)\, g(x) + f(x) \frac{\partial g}{\partial x_i}(x)$$

$$\frac{\partial^2}{\partial x_i^2}(f \cdot g)(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}(x)\, g(x) + f(x) \frac{\partial g}{\partial x_i}(x) \right)$$

$$= \frac{\partial^2 f}{\partial x_i^2}(x)\, g(x) + 2 \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) + f(x) \frac{\partial^2 g}{\partial x_i^2}(x)$$

Thus, $\Delta(f\cdot g) = (\Delta f)\cdot g + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle + f\cdot(\Delta g)$

Hence, if $f, g \in H(\Omega)$, then

$$\Delta(f\cdot g) = 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle,$$

i.e., $f\cdot g \in H(\Omega) \iff \langle \operatorname{grad} f, \operatorname{grad} g\rangle \equiv 0$ on Ω.

Assignment 1 B

Exercise 1

$f \in \mathcal{O}(\Omega)$ without zeros.

Take $z_0 \in \Omega$ and $r > 0$ such that $D(z_0, r) \subset \Omega$.

Then, there exists $h \in \mathcal{O}(D(z_0, r))$ such that

$$f(z) = \exp(h(z)) \qquad \forall\, z \in D(z_0, r)$$

$$\Longrightarrow \quad |f(z)| = \exp(\operatorname{Re}(h(z))) \quad \forall\, z \in D(z_0, r)$$

$\Rightarrow \quad u(z) = \log|f(z)| = \operatorname{Re}(h(z)) \quad \forall z \in D(z_0, r)$

i.e., $u|_{D(z_0,r)} \in H(D(z_0,r))$.

Hence, $u \in H(\Omega)$.

Exercise 2

(i) $\mathcal{M}(u; x_0, r_1) - \mathcal{M}(u; x_0, r_2) \qquad (r_1, r_2 \in (0, r_0))$

$$= \frac{1}{\omega_N N} \int_{S^{N-1}} \left[u(x_0 + r_1 \varsigma) - u(x_0 + r_2 \varsigma) \right] d\sigma^{N-1}(\varsigma)$$

Since u is uniformly continuous on

$\overline{B(0,r_0')}$ for any fixed $0 < r_0' < r_0$,

we see that

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x_1, x_2 \in \overline{B(0,r_0')}$:

$$|x_1 - x_2| < \delta \Rightarrow |u(x_1) - u(x_2)| < \varepsilon .$$

Take $r_1, r_2 \in (0, r_0')$ with $|r_1 - r_2| < \delta$, then

$$\max_{\varsigma \in S^{N-1}} |u(\underbrace{x_0 + r_1\varsigma}_{= x_1}) - u(\underbrace{x_0 + r_2\varsigma}_{= x_2})| < \varepsilon$$

$|x_1 - x_2| = |r_1 - r_2| < \delta$

$$\Rightarrow \quad |\mathcal{M}(u; x_0, r_1) - \mathcal{M}(u; x_0, r_2)| < \varepsilon$$

Thus, $\mathcal{M}(u; x_0, \cdot)$ is uniformly continuous on $[0, r_0')$ for each $0 < r_0' < r_0$; therefore, $\mathcal{M}(u; x_0, \cdot)$ is continuous on $(0, r_0)$.

Similarly, it follows that $\mathcal{A}(u; x_0, \cdot)$ is continuous on $(0, r_0)$.

Moreover, since u is continuous in 0, we have

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \Omega :$

$$|x - x_0| < \delta \implies |u(x) - u(x_0)| < \varepsilon$$

Then, for all $r \in (0, r_0)$ with $r < \delta$:

$$\mathcal{M}(u; x_0, r) - u(x_0) = \frac{1}{\omega_N N} \int_{S^{N-1}} [u(x_0 + r\xi) - u(x_0)] \, d\sigma^{N-1}(\xi)$$

$$|\mathcal{M}(u; x_0, r) - u(x_0)| \leq \max_{\xi \in S^{N-1}} |u(\underbrace{x_0 + r\xi}_{= x}) - u(x_0)| < \varepsilon$$

$|x - x_0| = r < \delta$

Thus, $\lim_{r \downarrow 0} \mathcal{M}(u; x_0, r) = u(x_0)$.

Similarly, $\lim_{r \downarrow 0} A(u; x_0, r) = u(x_0)$.

(ii) Recall that

$$\underbrace{\int_{B(x_0,r)} u(x)\, d\lambda^N(x)}_{= \omega_N r^N A(u; x_0, r)} = \int_0^r \rho^{N-1} \underbrace{\int_{S^{N-1}} u(x_0 + \rho \varsigma)\, d\sigma^{N-1}(\varsigma)}_{= N\omega_N \mathcal{M}(u; x_0, \rho)}\, d\rho$$

$$\Rightarrow \quad r^N A(u; x_0, r) = N \int_0^r \rho^{N-1} \mathcal{M}(u; x_0, \rho)\, d\rho \qquad (*)$$

for all $r \in [0, r_0)$

* Suppose that $\mathcal{M}(u; x_0, r) = u(x_0) \quad \forall r \in (0, r_0)$

$$\Rightarrow \quad r^N \mathcal{A}(u; x_0, r) \overset{(*)}{=} N \underbrace{\int_0^r \rho^{N-1} \, d\rho}_{= \frac{1}{N} r^N} \cdot u(x_0)$$

$$\Rightarrow \quad \mathcal{A}(u; x_0, r) = u(x_0) \qquad \forall r \in (0, r_0)$$

* Suppose that $\mathcal{A}(u; x_0, r) = u(x_0) \quad \forall r \in (0, r_0)$

$$\Rightarrow \quad \underbrace{\frac{d}{dr}\left(r^N \mathcal{A}(u; x_0, r)\right)}_{= N r^{N-1} u(x_0)} \overset{(*)}{=} N r^{N-1} \mathcal{M}(u; x_0, r)$$

$$\Longrightarrow \quad \mathcal{M}(u; x_0, r) = u(x_0) \qquad \forall r \in (0, r_0).$$

Note on the integration formula

$$(*) \int_{B(x_0,r)} f(x)\, d\lambda^N(x) = \int_0^r \rho^{N-1} \int_{\mathbb{S}^{N-1}} f(x_0 + r\varsigma)\, d\sigma^{N-1}(\varsigma)\, d\rho$$

Def.: $\Phi : \mathbb{R}_+ \times \mathbb{S}^{N-1} \longrightarrow \mathbb{R}^N, \quad (r, \varsigma) \longmapsto r\varsigma$

Claim: μ^N measure on $\mathbb{R}_+ \times \mathbb{S}^{N-1}$ s.t. $\dfrac{d\mu^N}{d(\lambda^1 \times \sigma^{N-1})}(r, \varsigma) = r^{N-1}$

Then $\Phi_* \mu^N = \lambda^N$.

⌜This implies (*), because (w.l.g $x_0 = 0$)

$$\int_{B(0,r)} f(x)\, d\lambda^N(x) = \int_{[0,r)\times S^{N-1}} f(\rho\varsigma)\, d\mu^N(\rho,\varsigma)$$

$$= \int_{[0,r)\times S^{N-1}} f(\rho\varsigma)\rho^{N-1}\, d\lambda^1(\rho)\, d\sigma^{N-1}(\varsigma)$$

$$= \int_0^r \rho^{N-1} \int_{S^{N-1}} f(\rho\varsigma)\, d\sigma^{N-1}(\varsigma)\, d\rho$$

⌞

Consider $B := \Phi([r_1, r_2] \times A)$, $A \subseteq S^{N-1}$. Then

$$(\Phi_* \mu^N)(B) = \mu^N([r_1, r_2] \times A) = \int_{r_1}^{r_2} \rho^{N-1}\, d\rho \cdot \sigma^{N-1}(A)$$

$$= \frac{1}{N}\left(r_2^N - r_1^N\right)\sigma^{N-1}(A)$$

$$= r_2^N \lambda^N(\tilde{A}) - r_1^N \lambda^N(\tilde{A}), \quad \tilde{A} := \{ta \mid t \in [0,1], a \in A\}$$

$$= \lambda^N\left(\{ta \mid t \in [r_1, r_2], a \in A\}\right)$$

$$= \lambda^N(B).$$