UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 1 B
for the first tutorial on *Monday, May 18, 1:00 pm*

Exercise 1. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be an open subset. Suppose that $f : \Omega \to \mathbb{C}$ is a holomorphic function which is nowhere vanishing, i.e., it satisfies $f(z) \neq 0$ for all $z \in \Omega$. Prove that

$$u: \ \Omega \longrightarrow \mathbb{R}, \quad z \longmapsto \log |f(z)|$$

is a harmonic function.

Hint: One can show this either by brute force or, in a more clever way, by using some facts from complex analysis about logarithms of nowhere vanishing holomorphic functions.

Exercise 2. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be an open subset and consider a function $u \in C(\Omega)$. Further, let $x_0 \in \Omega$ and $r_0 > 0$ be given such that $B(x_0, r_0) \subseteq \Omega$ holds. For every $0 < r < r_0$, we consider the means $\mathcal{M}(u; x_0, r)$ and $\mathcal{A}(u; x_0, r)$ as introduced in Definition 3.2 of the lecture. Prove the following assertions:

(i) The functions

$$\begin{aligned} \mathcal{M}(u; x_0, \cdot): \ (0, r_0) &\longrightarrow \mathbb{R}, \quad r \longmapsto \mathcal{M}(u; x_0, r), \\ \mathcal{A}(u; x_0, \cdot): \ (0, r_0) &\longrightarrow \mathbb{R}, \quad r \longmapsto \mathcal{A}(u; x_0, r) \end{aligned}$$

are continuous and we have that

$$\lim_{r \searrow 0} \mathcal{M}(u; x_0, r) = u(x_0) \qquad \text{and} \qquad \lim_{r \searrow 0} \mathcal{A}(u; x_0, r) = u(x_0).$$

(ii) For all $r \in (0, r_0)$, it holds true that

$$r^N \mathcal{A}(u; x_0, r) = N \int_0^r \rho^{N-1} \mathcal{M}(h; x_0, \rho) \, \mathrm{d}\rho.$$

Use this to show that $\mathcal{M}(u; x_0, r) = u(x_0)$ holds for all $r \in (0, r_0)$ if and only if $\mathcal{A}(u; x_0, r) = u(x_0)$ holds for all $r \in (0, r_0)$.