Assignments for the lecture

*Potential Theory in the Complex Plane*

Summer term 2020

Assignment 2 A

for the tutorial on *Monday, June 8 (!), 1:00 pm*

---

**Exercise 1.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. Suppose that $(u_n)_{n \in \mathbb{N}}$ is a sequence in $H(\Omega)$ which converges locally uniformly on $\Omega$ (i.e., uniformly on each compact subset of $\Omega$) to some function $u : \Omega \rightarrow \mathbb{R}$. Prove that $u \in H(\Omega)$.

**Exercise 2.** For any point $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we set $x' := (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\overline{x} := (x', -x_N) \in \mathbb{R}^N$. We write $\mathbb{R}^N$ as the disjoint union $\mathbb{R}^N = \mathbb{R}_+^N \cup \mathbb{R}_0^N \cup \mathbb{R}_-^N$, where $\mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N > 0\}$, $\mathbb{R}_-^N := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N < 0\}$, and $\mathbb{R}_0^N := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N = 0\}$. Note that, for every $x \in \mathbb{R}^N$, $\overline{x}$ is the image of $x$ under reflection in the hyperplane $\mathbb{R}_0^N$.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be an open subset which is symmetric under reflection in $\mathbb{R}_0^N$, i.e., we have $\overline{x} \in \Omega$ for every $x \in \Omega$. We set $\Omega_+ := \Omega \cap \mathbb{R}_+^N$, $\Omega_- := \Omega \cap \mathbb{R}_-^N$, and $\Omega_0 := \Omega \cap \mathbb{R}_0^N$.

Suppose that $u \in H(\Omega_+)$ satisfies $\lim_{\Omega_+ \ni x \to x_0} u(x) = 0$ for every $x_0 \in \Omega_0$. Show that

$$
\hat{u} : \Omega \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 
\quad u(x), & \text{if } x \in \Omega_+ \\
\quad 0, & \text{if } x \in \Omega_0 \\
\quad -u(\overline{x}), & \text{if } x \in \Omega_-
\end{cases}
$$

yields a well-defined function which is harmonic on $\Omega$. 