UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 2 A
for the tutorial on *Monday, June 8 (!), 1:00 pm*

Exercise 1. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H(\Omega)$ which *converges locally uniformly on* Ω (i.e., uniformly on each compact subset of Ω) to some function $u : \Omega \to \mathbb{R}$. Prove that $u \in H(\Omega)$.

Exercise 2. For any point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we set $x' := (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\overline{x} := (x', -x_N) \in \mathbb{R}^N$. We write $\mathbb{R}^N$ as the disjoint union $\mathbb{R}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_0 \cup \mathbb{R}^N_-$, where $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N > 0\}$, $\mathbb{R}^N_- := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N < 0\}$, and $\mathbb{R}^N_0 := \{x = (x', x_N) \in \mathbb{R}^N \mid x_N = 0\}$. Note that, for every $x \in \mathbb{R}^N$, $\overline{x}$ is the image of x under reflection in the hyperplane $\mathbb{R}^N_0$.
Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be an open subset which is symmetric under reflection in $\mathbb{R}^N_0$, i.e., we have $\overline{x} \in \Omega$ for every $x \in \Omega$. We set $\Omega_+ := \Omega \cap \mathbb{R}^N_+$, $\Omega_- := \Omega \cap \mathbb{R}^N_-$, and $\Omega_0 := \Omega \cap \mathbb{R}^N_0$. Suppose that $u \in H(\Omega_+)$ satisfies $\lim_{\Omega_+ \ni x \to x_0} u(x) = 0$ for every $x_0 \in \Omega_0$. Show that

$$\tilde{u} :\ \Omega \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} u(x), & \text{if } x \in \Omega_+ \\ 0, & \text{if } x \in \Omega_0 \\ -u(\overline{x}), & \text{if } x \in \Omega_- \end{cases}$$

yields a well-defined function which is harmonic on Ω.