# Assignment 2 A

## Exercise 1

Let $x_0 \in \Omega$ and $r > 0$ with the property that $\overline{B(x_0,r)} \subset \Omega$ be given. Since $\overline{B(x_0,r)}$ is compact, we have by assumption

$$\max_{x \in \overline{B(x_0,r)}} |u_n(x) - u(x)| \longrightarrow 0 \quad \text{as } n \to \infty .$$

We infer that $u|_{\overline{B(x_0,r)}} \in C(\overline{B(x_0,r)})$ and further

$$|A(u_n; x_0, r) - A(u; x_0, r)| \leq \frac{1}{\omega_N r^N} \int_{\overline{B(x_0,r)}} |u_n(x) - u(x)| \, d\lambda^N(x)$$

$$\leq \max_{x \in B(x_0, r)} |u_n(x) - u(x)| \longrightarrow 0.$$

On the other hand, we have $u_n(x_0) \to u(x_0)$ as $n \to \infty$ since $\{x_0\}$ is compact. Therefore, by the MVP of $u_n$,

$$\begin{array}{ccc} A(u_n; x_0, r) & = & u_n(x_0) \\ \downarrow & & \downarrow \\ A(u; x_0, r) & & u(x_0) \end{array} \quad \text{as } n \to \infty,$$

hence $A(u; x_0, r) = u(x_0)$. Thus, $u \in C(\Omega)$ and $u$ has the MVP on $\Omega$, which implies $u \in H(\Omega)$, as desired.

## Exercise 2

It suffices to show:

$$\forall x_0 \in \Omega \quad \exists r_0 > 0 : \quad B(x_0, r_0) \subset \Omega \text{ and}$$

$$\Omega = \Omega_+ \cup \Omega_0 \cup \Omega_- \qquad \forall r \in (0, r_0) : \quad u(x_0) = A(u; x_0, r).$$

**Case 1:** $x_0 \in \Omega_+$

Choose $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega_+$. Then

$$A(\tilde{u}; x_0, r) = A(u; x_0, r) = u(x_0) = \tilde{u}(x_0)$$

for all $r \in (0, r_0)$, since $u$ has the MVP on $\Omega_+$.

Case 2: $x_0 \in \Omega_-$

Choose $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega_-$. Then

$$A(\tilde{u}; x_0, r) = -\frac{1}{\omega_N r^N} \int_{\overline{B(x_0,r)}} u(\bar{x})\, d\lambda^N(x)$$

transformation formula for $x \mapsto \bar{x}$, which maps $\overline{B(x_0,r)}$ onto $\overline{B(\bar{x}_0,r)}$.

$$= -\frac{1}{\omega_N r^N} \int_{\overline{B(\bar{x}_0,r)}} u(x)\, d\lambda^N(x)$$

$$= -A(u; \bar{x}_0, r) = -u(\bar{x}_0) = \tilde{u}(x_0)$$

for all $r \in (0, r_0)$, since $u$ has the MVP on $\Omega$.

Case 3: $x_0 \in \Omega_0$

Choose any $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega$. Then

$$A(\tilde{u}; x_0, r) = \frac{1}{\omega_N r^N} \int_{\overline{B(x_0,r)}} \tilde{u}(x)\, d\lambda^N(x)$$

$\boxed{B_\pm := B(x_0, r) \cap \mathbb{R}^N_\pm}$

$$= \frac{1}{\omega_N r^N} \left[ \int_{B_+} \tilde{u}(x)\, d\lambda^N(x) + \int_{B_-} \tilde{u}(x)\, d\lambda^N(x) \right].$$

Now, $\int_{B_-} \tilde{u}(x)\, d\lambda^N(x) = -\int_{B_-} u(\bar{x})\, d\lambda^N(x) = -\int_{B_+} u(x)\, d\lambda^N(x)$

transformation formula for $x \mapsto \bar{x}$, which sends $B_-$ to $B_+$.

Hence, $A(\tilde{u}; x_0, r) = 0 = \tilde{u}(x_0)$.

# Assignment 2B

## Exercise 1

We consider the Poisson kernel $K_{x_0,r}$ on $B(x_0,r)$.

Let $y \in \partial B(x_0,r)$ be fixed. Then

$$K_{x_0,r}(\cdot,y) = \frac{1}{N\omega_N r} u \cdot v, \quad \text{where}$$

$$u(x) = r^2 - \|x-x_0\|^2 \quad \text{and} \quad v(x) = \|x-y\|^{-N}.$$

We compute that $\operatorname{grad} u(x) = -2(x-x_0)$ and $\Delta u(x) = -2N$.

Further, since $v(x) = \left( \sum_{j=1}^{N} (x_j - y_j)^2 \right)^{-N/2}$, we get

$$\operatorname{grad} v(x) = -N \frac{x-y}{\|x-y\|^{N+2}} \quad \text{and} \quad \Delta v(x) = 2N \frac{1}{\|x-y\|^{N+2}}.$$

Note:
$$\frac{\partial}{\partial x_i} \left( -N \left( \sum_{j=1}^{N} (x_j - y_j)^2 \right)^{-\frac{N+2}{2}} (x_i - y_i) \right)$$

$$= N \cdot \frac{N+2}{\cancel{2}} \cdot \cancel{2} \cdot (x_i - y_i) \cdot \left( \sum_{j=1}^{N} (x_j - y_j)^2 \right)^{-\frac{N+2}{2} - 1} (x_i - y_i)$$

$$- N \left( \sum_{j=1}^{N} (x_j - y_j)^2 \right)^{-\frac{N+2}{2}}$$

$$= N \frac{1}{\|x-y\|^{N+2}} \left( (N+2) \frac{(x_i - y_i)^2}{\|x-y\|^2} - 1 \right)$$

and hence $\Delta v(x) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\cdots) = N \frac{1}{\|x-y\|^{N+2}}\left((N+2)-N\right)$

$$= 2N \frac{1}{\|x-y\|^{N+2}} .$$

We conclude that

$$\Delta(uv)(x) = -2N \frac{1}{\|x-y\|^{N}} + 2N \frac{r^2 - \|x-x_0\|^2}{\|x-y\|^{N+2}}$$

$$+ 4N \frac{\langle x-x_0, x-y\rangle}{\|x-y\|^{N+2}}$$

$$= -2N \|x-y\|^{-(N+2)}\left(\|x-y\|^2 - r^2 + \|x-x_0\|^2 - 2\langle x-x_0, x-y\rangle\right)$$

We observe that

$$r^2 = \|y - x_0\|^2 = \|(x-x_0)-(x-y)\|^2$$

$$= \|x-x_0\|^2 + \|x-y\|^2 - 2\langle x-x_0, x-y\rangle$$

Hence, $\Delta(u \cdot v) \equiv 0$ and, in particular, $\Delta K_{x_0,r}(\cdot, y) \equiv 0$.

## Exercise 2:

(i) Since $\overline{D(z_0,r)} \subset \Omega$, we find $r_1 > r$ such that $D(z_0, r_1) \subseteq \Omega$. Further, since $|w| < r$ for

$w := z - z_0$ and any fixed $z \in D(z_0, r)$, we see

that $\zeta \mapsto r^2 - \zeta \overline{w}$ vanishes nowhere on $D(z_0, r_2)$

with $r_2 := \dfrac{r^2}{|w|} > r$. Put $r_0 := \min\{r_1, r_2\}$. Then

$(r_2 = \infty \text{ if } w = 0)$

$$F : D(z_0, r_0) \to \mathbb{C}, \quad \zeta \mapsto \frac{r^2 - |w|^2}{r^2 - \zeta \overline{w}} f(z_0 + \zeta)$$

is well-defined and holomorphic.

Next, we compute that

$$\int_0^{2\pi} K_{z_0, r}(z, z_0 + re^{it}) f(z_0 + re^{it}) \, r \, dt$$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} \underbrace{\frac{r^2 - |z - z_0|^2}{|z - (z_0 + re^{it})|^2}} f(z_0 + re^{it})\, dt$$

$$= \frac{r^2 - |w|^2}{|w - re^{it}|^2} = \frac{r^2 - |w|^2}{(w - re^{it})(\overline{w} - re^{-it})}$$

$$= \frac{(r^2 - |w|^2) re^{it}}{(w - re^{it})(\overline{w} re^{it} - r^2)}$$

$$= \frac{1}{re^{it} - w} \cdot \frac{r^2 - |w|^2}{r^2 - \overline{w} re^{it}}\, re^{it}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{F(re^{it})}{re^{it} - w}\, re^{it}\, dt = \frac{1}{2\pi i} \int_{\gamma_{0,r,\sigma}} \frac{F(\zeta)}{\zeta - w}\, d\zeta$$

$$= F(w) = \frac{r^2 - |w|^2}{r^2 - \bar{w} w} f(z_0 + w) = f(z).$$

(ii) Consider $r_0 > r$ such that $D(z_0, r_0) \subset \Omega$. Then, there exists $f \in \mathcal{O}(D(z_0, r_0))$ such that $u = \mathrm{Re}(f)$.

By (i), we know that

$$f(z) = \int_0^{2\pi} K_{z_0, r}(z, z_0 + re^{it}) f(z_0 + re^{it}) r \, dt$$

from which we deduce (since $K_{z_0,r}$ is real-valued)

$$u(z) = \mathrm{Re}(f(z)) = \int_0^{2\pi} K_{z_0,r}(z, z_0 + re^{it}) u(z_0 + re^{it}) r \, dt.$$