UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 2 B
for the tutorial on *Monday, June 8 (!), 1:00 pm*

Exercise 1. For $x_0 \in \mathbb{R}^N$ and $r > 0$, consider the Poisson kernel of $B(x_0, r)$, namely

$$K_{x_0,r} : \; B(x_0,r) \times \partial B(x_0,r) \longrightarrow \mathbb{R}, \quad K_{x_0,r}(x,y) = \frac{1}{N\omega_N r} \frac{r^2 - \|x - x_0\|^2}{\|x-y\|^N}.$$

Prove that for each fixed $y \in \partial B(x_0, r)$, the function

$$K_{x_0,r}(\cdot, y) : \; B(x_0,r) \longrightarrow \mathbb{R}, \quad x \longmapsto K_{x_0,r}(x,y)$$

is harmonic on $B(x_0, r)$.

Hint: Write $K_{x_0,r}(\cdot, y)$ as the product of two smooth functions $u, v : B(x_0, r) \to \mathbb{R}$. Then, use the identity $\Delta(uv) = v\Delta u + u\Delta v + 2\langle \operatorname{grad} u, \operatorname{grad} v\rangle$ in order to show that $\Delta K_{x_0,r}(\cdot, y) \equiv 0$.

Exercise 2. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be an open subset and let $z_0 \in \Omega$ and $r > 0$ be given such that $\overline{D(z_0, r)} \subset \Omega$. Our goal is to deduce Poisson's integral formula from Cauchy's integral formula. We proceed as follows:

(i) Prove that every $f \in \mathcal{O}(\Omega)$ satisfies for all $z \in D(z_0, r)$

$$f(z) = \int_0^{2\pi} K_{z_0,r}(z, z_0 + re^{it}) f(z_0 + re^{it})\, r \,\mathrm{d}t.$$

Hint: For fixed $z \in D(z_0, r)$, put $w := z - z_0$ and verify that $F(\zeta) := \frac{r^2 - |w|^2}{r^2 - \zeta\overline{w}} f(z_0 + \zeta)$ defines a function which is holomorphic on a neighborhood of $\overline{D(0, r)}$. Show that the integral on the right hand side of the formula asserted in (i) can be rewritten as $\frac{1}{2\pi i} \int_{\gamma_{0,r,\circlearrowleft}} \frac{1}{\zeta - w} F(\zeta)\, \mathrm{d}\zeta$ and apply Cauchy's integral formula.

(ii) Deduce from (i) that every $u \in H(\Omega)$ satisfies for all $z \in D(z_0, r)$

$$u(z) = \int_0^{2\pi} K_{z_0,r}(z, z_0 + re^{it}) u(z_0 + re^{it})\, r \,\mathrm{d}t.$$