UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 3 A
for the tutorial on *Monday, June 22, 1:00 pm*

Exercise 1. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and connected. We denote by $H_+(\Omega)$ the set of all harmonic functions $u : \Omega \to \mathbb{R}$ which satisfy $u(x) \geq 0$ for all $x \in \Omega$.
Prove the following assertion: for each choice of $x, y \in \Omega$, there exists a constant $\tau > 0$ (depending only on x and y) such that

$$\tau^{-1} u(x) \leq u(y) \leq \tau u(x) \qquad \text{for all } u \in H_+(\Omega).$$

Hint: Check that one obtains an equivalence relation $\sim$ on Ω by the following rule: for $x, y \in \Omega$, one has $x \sim y$ if and only if there exists some $\tau > 0$ such that $\tau^{-1} u(x) \leq u(y) \leq \tau u(x)$ holds for every $u \in H_+(\Omega)$. Deduce from Harnack's inequalities that the equivalence class $\{y \in \Omega \mid x \sim y\}$ of any $x \in \Omega$ is an open subset of Ω. Finally, use that Ω is connected in order to show that there is only one such equivalence class.

Exercise 2. Let X be a topological space and let $f : X \to [-\infty, \infty)$ be an upper semicontinuous function (i.e., $f^{-1}([-\infty, a))$ is an open subset of X for each $a \in \mathbb{R}$).

(i) Show that for every limit point $y \in X$,

$$\limsup_{x \to y} f(x) \leq f(y).$$

Hint: Recall that by definition

$$\limsup_{x \to y} f(x) = \inf_{U \in \mathcal{U}(y)} \sup_{x \in U \setminus \{y\}} f(x),$$

where $\mathcal{U}(y)$ denotes the set of all neighborhoods of y.

(ii) Suppose that $K \subseteq X$ is a compact subset. Show that

$$\sup_{x \in K} f(x) < \infty$$

and that there exists some $x_0 \in K$ such that $f(x_0) = \sup_{x \in K} f(x)$.