Assignment 3 A

Exercise 1

We define $\sim$ by

$$x \sim y \;:\Longleftrightarrow\; \exists \tau > 0 \;\forall u \in H_+(\Omega) : \tau^{-1} u(x) \leq u(y) \leq \tau u(x)$$

We see that $\sim$ forms an equivalence relation on Ω, i.e.,

- $\sim$ is symmetric; indeed,

$$\Longrightarrow \; u(x) \leq \tau u(y)$$

$$\tau^{-1} u(x) \leq u(y) \leq \tau u(x)$$

$$\Longrightarrow \; \tau^{-1} u(y) \leq u(x)$$

$$\tau^{-1} u(y) \leq u(x) \leq \tau u(y)$$

- $\sim$ is reflexive (obvious, take $\tau = 1$)
- $\sim$ is transitive; indeed,

$$\Rightarrow u(x) \leq (\tau_1 \tau_2) u(z)$$

$$x \sim y \Rightarrow \exists \tau_1 > 0 \ \forall u \in H_+(\Omega): \ \tau_1^{-1} u(y) \leq \overbrace{u(x) \leq \tau_1 u(y)}$$

$$y \sim z \Rightarrow \exists \tau_2 > 0 \ \forall u \in H_+(\Omega): \ \underbrace{\tau_2^{-1} u(z) \leq u(y)} \leq \tau_2 u(z)$$

$$\Rightarrow (\tau_1 \tau_2)^{-1} u(z) \leq u(x)$$

so that we get $x \sim z$ (with $\tau := \tau_1 \cdot \tau_2$).

Fix $x \in \Omega$ and set $\Omega_0 := \{ y \in \Omega \mid x \sim y \}$. Then:

- $\Omega_0 \neq \emptyset$ as $x \in \Omega_0$;

- Ω_0 is open. Indeed, take $y \in \Omega_0$ and choose $r > 0$ such that $B(y,r) \subset \Omega$, then Harnack's inequalities give that for each $z \in B(y,r)$

$$\sigma_1 u(y) \le u(z) \le \sigma_2 u(y) \qquad \forall u \in H_+(\Omega),$$

where $\sigma_1 := \dfrac{(r - \|z-y\|) r^{N-2}}{(r + \|z-y\|)^{N-1}}$, $\sigma_2 := \dfrac{(r + \|z-y\|) r^{N-2}}{(r - \|z-y\|)^{N-1}}$

Choose $\tau \ge \max\{\sigma_1^{-1}, \sigma_2\}$; then $\tau \ge \sigma_2$ and $\tau^{-1} \le \sigma_1$ and thus

$$\tau^{-1} u(y) \le u(z) \le \tau u(y) \qquad \forall u \in H_+(\Omega),$$

i.e., $y \sim z$. Then $x \sim y$ and $y \sim z$, hence $x \sim z$, i.e., $z \in \Omega_0$. Therefore, $B(y,r) \subseteq \Omega_0$.

- $\Omega \setminus \Omega_0$ is open. Indeed, take $z \in \Omega \setminus \Omega_0$ and choose $r > 0$ such that $B(z,r) \subseteq \Omega$. By Harnack's inequality, we see that $y \sim z$ for each $y \in B(z,r)$. Thus, $B(z,r) \subseteq \Omega \setminus \Omega_0$ because otherwise there would be $y \in B(z,r)$ such that $x \sim y$ and hence $x \sim z$ in contradiction to $z \notin \Omega_0$.

Since Ω is connected, it follows that $\Omega_0 = \Omega$, as desired.

Exercise 2

(i) Note that

$$\limsup_{x \to y} f(x) = \inf_{U \in \mathcal{U}(y)} \sup_{x \in U \setminus \{y\}} f(x) \leq f(y)$$

$$\Longleftrightarrow \forall \varepsilon > 0 \ \exists U \in \mathcal{U}(y) \ \forall x \in U \setminus \{y\}: \ f(x) \leq f(y) + \varepsilon .$$

Let $\varepsilon > 0$. Since f is usc, we know that

$$U := f^{-1}([-\infty, f(y) + \varepsilon))$$

is open in X. Because also $y \in U$, it follows $U \in \mathcal{U}(y)$.

Further, we have

$$\forall x \in U: \quad f(x) \in [-\infty, f(y) + \varepsilon),$$

$$\text{i.e.,} \quad f(x) \leq f(y) + \varepsilon.$$

(ii) For $a \in \mathbb{R}$, we set $U_a := f^{-1}([-\infty, a))$. Then $(U_a)_{a \in \mathbb{R}}$ forms an open cover of K. By compactness of K, there is a finite sub-cover $(U_{a_k})_{k=1}^{n}$; put $a := \max\{a_1, \ldots, a_n\}$. Then $K \subseteq \bigcup_{k=1}^{n} U_{a_k} = U_a$, hence $f(K) \subseteq [-\infty, a)$. Thus, $\sup_{x \in K} f(x) \leq a < \infty$.

Assume that there is <u>no</u> $x_0 \in K$ such that

$$f(x_0) = \sup_{x \in K} f(x) =: a_0 .$$

Then $(U_a)_{a \in (-\infty, a_0)}$ is an open cover of K with <u>no</u> finite sub-cover, in contradiction to compactness of K.

Assignment 3B:

Exercise 1

(i) Let $f := \max\{f_1, f_2\}$. Then, for every $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in X \mid f_1(x) < a \wedge f_2(x) < a\}$$

$$= \{x \in X \mid f_1(x) < a\} \cap \{x \in X \mid f_2(x) < a\}$$

$$= \underbrace{f_1^{-1}([-\infty, a))}_{\text{open } (f_1 \text{ usc})} \cap \underbrace{f_2^{-1}([-\infty, a))}_{\text{open } (f_2 \text{ usc})}$$

is open. Thus, f is usc.

Let $f := \alpha_1 f_1 + \alpha_2 f_2$. Then, for every $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in X \mid \alpha_1 f_1(x) + \alpha_2 f_2(x) < a\}$$

$$= \{x \in X \mid \exists\, a_1, a_2 \in \mathbb{R},\ \alpha_1 a_1 + \alpha_2 a_2 \leq a : f_1(x) < a_1 \wedge f_2(x) < a_2\}$$

$$= \bigcup_{\substack{a_1, a_2 \in \mathbb{R} \\ \alpha_1 a_1 + \alpha_2 a_2 \leq a}} \underbrace{f_1^{-1}([-\infty, a_1))}_{\text{open}} \cap \underbrace{f_2^{-1}([-\infty, a_2))}_{\text{open}}$$

is open. Thus, f is usc.

(ii) Let $s := \max\{s_1, s_2\}$. Then s is usc by (i), $s \not\equiv -\infty$ on Ω, and satisfies for every $\overline{B(x,r)} \subset \Omega$

$$\left.\begin{aligned} s_1(x) &\le \mathcal{M}(s_1;x,r) \le \mathcal{M}(s;x,r) \\ s_2(x) &\le \mathcal{M}(s_2;x,r) \le \mathcal{M}(s;x,r) \end{aligned}\right\} s(x) \le \mathcal{M}(s;x,r).$$

Thus, $s \in \mathcal{S}(\Omega)$.

Let $s := \alpha_1 s_1 + \alpha_2 s_2$. Then s is usc by (i), $s \not\equiv -\infty$ on Ω (by Corollary 5.5), and satisfies for every $\overline{B(x,r)} \subset \Omega$

$$\begin{aligned} s(x) &= \alpha_1 s_1(x) + \alpha_2 s_2(x) \\ &\le \alpha_1 \mathcal{M}(s_1;x,r) + \alpha_2 \mathcal{M}(s_2;x,r) \\ &= \mathcal{M}(\alpha_1 s_1 + \alpha_2 s_2;x,r) = \mathcal{M}(s;x,r). \end{aligned}$$

Thus, $s \in \mathcal{S}(\Omega)$.

Exercise 2:

① s is USC

Let $a \in \mathbb{R}$ be given. Then

$$s^{-1}([-\infty, a)) = \{x \in X \mid \inf_{n \in \mathbb{N}} s_n(x) < a\}$$

$$= \{x \in X \mid \exists n \in \mathbb{N} : s_n(x) < a\}$$

$$= \bigcup_{n \in \mathbb{N}} s_n^{-1}([-\infty, a))$$

is open.

② $s(x) \leq \mathcal{M}(s;x,r)$ whenever $\overline{B(x,r)} \subset \Omega$

Put $a := \sup_{y \in \partial B(x,r)} s_1(y)$; note that $a < \infty$ as s_1 is usc

Define $f_n := a - s_n : \Omega \to [0,+\infty]$. Then

- $\forall n \in \mathbb{N}: \quad f_n(x) \leq f_{n+1}(x)$
- $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in \Omega,$

where $f := a - s : \Omega \to [0,+\infty]$.

Then, by the monotone convergence theorem,

$$\lim_{n\to\infty} \mathcal{M}(f_n; x, r) = \mathcal{M}(f; x, r).$$

Thus,

$$\left.\begin{aligned}\underbrace{\lim_{n\to\infty} \underbrace{\mathcal{M}(s_n; x, r)}_{\geq s_n(x)}}_{\geq \lim_{n\to\infty} s_n(x) = s(x)} = \mathcal{M}(s; x, r)\end{aligned}\right\} \; s(x) \leq \mathcal{M}(s; x, r).$$

③ If $s \not\equiv -\infty$, then $s \in S(\Omega)$ by ① and ②.