Assignment 4 A

Exercise 1:

We compute that

$$\frac{\partial}{\partial \bar{z}} s \circ f = \frac{\partial s}{\partial z} \circ f \cdot \cancelto{0}{\frac{\partial f}{\partial \bar{z}}} + \frac{\partial s}{\partial \bar{z}} \circ f \cdot \frac{\partial \bar{f}}{\partial \bar{z}} = \frac{\partial s}{\partial \bar{z}} \circ f \cdot \overline{\frac{\partial f}{\partial z}}$$

and then

$$\begin{aligned}\frac{\partial^2}{\partial z \partial \bar{z}} s \circ f &= \frac{\partial}{\partial z}\left(\frac{\partial s}{\partial \bar{z}} \circ f \cdot \overline{\frac{\partial f}{\partial z}} \right) \\ &= \frac{\partial}{\partial z}\left(\frac{\partial s}{\partial \bar{z}} \circ f \right) \cdot \overline{\frac{\partial f}{\partial z}} + \frac{\partial s}{\partial \bar{z}} \circ f \cdot \frac{\partial}{\partial z}\left(\overline{\frac{\partial f}{\partial z}} \right),\end{aligned}$$

where

$$\frac{\partial}{\partial z}\left(\frac{\partial s}{\partial \bar z}\circ f\right) = \frac{\partial^2 s}{\partial z \partial \bar z}\circ f \cdot \frac{\partial f}{\partial z} + \frac{\partial^2 s}{\partial \bar z^2}\circ f \cdot \cancel{\frac{\partial \bar f}{\partial z}} = \frac{1}{4}(\Delta s)\circ f \cdot \frac{\partial f}{\partial z}$$

$$\overline{\frac{\partial f}{\partial \bar z}} = 0$$

and

$$\frac{\partial}{\partial z}\left(\frac{\overline{\partial f}}{\partial z}\right) = \overline{\frac{\partial}{\partial \bar z}\left(\frac{\partial f}{\partial z}\right)} = \overline{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar z}\right)} = 0.$$

Thus,

$$\frac{1}{4}\Delta(s\circ f) = \frac{\partial^2}{\partial z \partial \bar z} s\circ f = \frac{1}{4}(\Delta s)\circ f \cdot \left|\frac{\partial f}{\partial z}\right|^2 = \frac{1}{4}\Delta s\circ f \cdot |f'|^2$$

$$\Longrightarrow \quad \Delta(s\circ f) = (\Delta s)\circ f \cdot |f'|^2$$

Exercise 2:

<u>Claim</u>: $\sup_{t \in T} f(x,t) < \infty \quad \forall x \in \Omega$

<u>Proof</u>: $K := \{x_0\} \times T \subset \Omega \times T$ is compact.

$$\overset{\text{Rem } 5.3(i)}{\underset{f \text{ usc}}{\Longrightarrow}} \quad \sup_{t \in T} f(x_0,t) = \sup_{(x,t) \in K} f(x,t) < \infty . \qquad \square$$

<u>Hence</u>: $s : \Omega \to [-\infty, +\infty),\ x \mapsto \sup_{t \in T} f(x,t)$

is well-defined.

<u>Claim:</u> S is USC.

<u>Proof:</u> Let $a \in \mathbb{R}$ be given. We want to show that

$$S^{-1}([-\infty, a)) \subseteq \Omega$$

is open. For this purpose, we take any

$$x_0 \in S^{-1}([-\infty, a)).$$

By definition of S, we have $f(x_0, t) < a$ for all $t \in T$, i.e., $(x_0, t) \in f^{-1}([-\infty, a))$ for all $t \in T$. Since f is USC, $f^{-1}([-\infty, a)) \subseteq \Omega \times T$ is open. Thus, we see:

$\forall t \in T \ \exists \varepsilon_t > 0,\ U_t \subseteq T$ open, $t \in U_t$:

$$(x_0, t) \in B(x_0, \varepsilon_t) \times U_t \subseteq f^{-1}([-\infty, a)).$$

Now, since $(U_t)_{t \in T}$ forms an open cover of T, we find a finite sub-cover $(U_{t_j})_{j=1}^{k}$, as T is compact.

Put $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_k\}$. Then, for all $x \in B(x_0, \varepsilon)$

$\forall t \in T \ \exists j \in \{1, \dots, k\}: \quad t \in U_{t_j}$ and hence

$(x, t) \in B(x_0, \varepsilon_{t_j}) \times U_{t_j} \subseteq f^{-1}([-\infty, a))$, i.e., $f(x,t) < a$.

By Remark 5.3 (i), $s(x) = \sup_{t \in T} f(x,t) < a$. Thus

$B(x_0,\varepsilon) \subseteq s^{-1}([-\infty,a))$; hence, $s^{-1}([-\infty,a))$ is open.

$\square$

<u>Claim:</u> s is subharmonic.

<u>Proof:</u> Let $\overline{B(x_0,r)} \subset \Omega$. Then

$$\mathcal{M}(s;x_0,r) = \frac{1}{N\omega_N}\int_{S^{N-1}} \underbrace{s(x_0+r\zeta)}_{\geq f(x_0+r\zeta,t)\ \forall t\in T}\, d\sigma^N(\zeta)$$

$$\geq \sup_{t\in T} \frac{1}{N\omega_N}\int_{S^{N-1}} f(x_0+r\zeta,t)\, d\sigma^N(\zeta)$$

$$= \sup_{t\in T} \mathcal{M}(f(\cdot,t);x_0,r)$$

$$\geq \sup_{t \in T} f(x_0, t)$$

$$= s(x_0).$$

$$\square$$

Assignment 4 B

Exercise 1:

Recall that by Gauss' divergence theorem

$$\int_\Omega \operatorname{div} F(x)\, d\lambda^N(x) = \int_{\partial\Omega} \langle F(x), n(x)\rangle \, d\sigma_{\partial\Omega}(x)$$

for all $F \in \left(C(\overline{\Omega}) \cap C^1(\Omega)\right)^N$.

Now, for $u, v \in C^2(\Omega_0)$, we find that

$$u \cdot \operatorname{grad} v \big|_{\overline{\Omega}} \quad \text{and} \quad v \cdot \operatorname{grad} u \big|_{\overline{\Omega}}$$

belong to $(C(\overline{\Omega}) \cap C^1(\Omega))^N$. Thus,

$$\int_{\Omega} \operatorname{div}(u \cdot \operatorname{grad} v)(x)\, d\lambda^N(x) = \int_{\partial\Omega} \underbrace{\langle u(x) \operatorname{grad} v(x), n(x) \rangle}\, d\sigma_{\partial\Omega}(x)$$

$$= u(x) \langle \operatorname{grad} v(x), n(x) \rangle$$

$$= u(x)\, D_n v(x)$$

$$\int_{\Omega} \operatorname{div}(v \cdot \operatorname{grad} u)(x)\, d\lambda^N(x) = \int_{\partial\Omega} \underbrace{\langle v(x) \operatorname{grad} u(x), n(x) \rangle}\, d\sigma_{\partial\Omega}(x)$$

$$= v(x) \langle \operatorname{grad} u(x), n(x) \rangle$$

$$= v(x)\, D_n u(x)$$

Further,

$$\operatorname{div}(u \cdot \operatorname{grad} v) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(u \cdot \frac{\partial v}{\partial x_j}\right)$$

$$= \sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_j} + u \cdot \frac{\partial^2 v}{\partial x_j^2}\right)$$

$$= \langle \operatorname{grad} u, \operatorname{grad} v \rangle + u \cdot \Delta v$$

and $\operatorname{div}(v \cdot \operatorname{grad} u) = \langle \operatorname{grad} v, \operatorname{grad} u \rangle + v \cdot \Delta u$,

so that

$$\operatorname{div}(u \cdot \operatorname{grad} v) - \operatorname{div}(v \cdot \operatorname{grad} u) = u \Delta v - v \Delta u.$$

Thus, in summary

$$\int_{\Omega} \big(u(x)\,\Delta v(x) - v(x)\,\Delta u(x) \big)\, d\lambda^n(x)$$

$$= \int_{\partial\Omega} \big(u(x)\, D_n v(x) - v(x)\, D_n u(x) \big)\, d\sigma_{\partial\Omega}(x).$$

Exercise 2:

(i) For $n \geq 2$, $w_1, \dots, w_n$, we define

$$V_n(w_1, \dots, w_n) := \prod_{1 \leq i < j \leq n} |w_i - w_j| \;;$$

thus,

$$\delta_n(K)^{\frac{n(n-1)}{2}} = \max_{(w_1,\dots,w_n)\in K^n} V_n(w_1,\dots,w_n).$$

Now, we notice that, for all $k = 1,\dots,n+1$,

$$V_{n+1}(w_1,\dots,w_{n+1}) = \prod_{1\le i<j\le n+1} |w_i - w_j|$$

$$= V_n(w_1,\dots,\check{w}_k,\dots,w_{n+1}) \cdot \prod_{\substack{\ell=1,\dots,n+1\\ \ell\neq k}} |w_\ell - w_k|.$$

Hence,

$$V_{n+1}(w_1,\dots,w_{n+1})^{n+1}$$

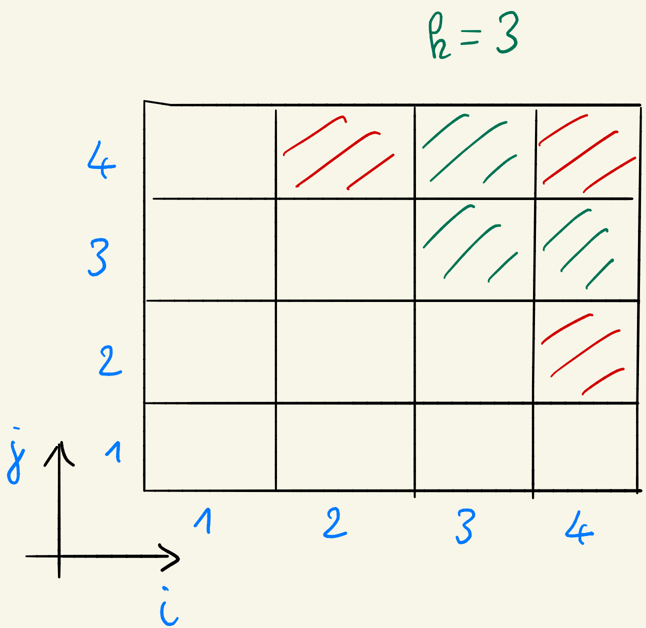
$$= \prod_{k=1}^{n+1} \underbrace{V_n(w_1,\dots,\check{w}_k,\dots,w_{n+1})}_{\leq \delta_n(K)^{\frac{n(n-1)}{2}}} \cdot \underbrace{\prod_{\substack{1\leq l,k\leq n+1 \\ l\neq k}} |w_l - w_k|}_{= V_{n+1}(w_1,\dots,w_{n+1})^2}$$

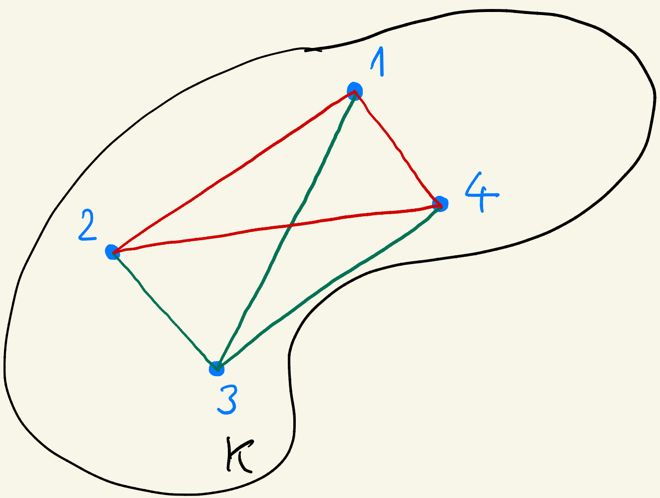
$$\Rightarrow V_{n+1}(w_1,\dots,w_{n+1}) \leq \delta_n(K)^{\frac{n(n+1)}{2}}$$

If we let $(w_1,\dots,w_{n+1})$ be a Fekete $(n+1)$-tuple for K, then $\delta_{n+1}(K)^{\frac{n(n+1)}{2}} = V_{n+1}(w_1,\dots,w_{n+1})$, so that the latter bound yields

$$\delta_{n+1}(K) \le \delta_n(K),$$

as desired.





(ii) We observe that for all $z \in K$

$$V_{n+1}(w_1,\dots,w_n,z) = V_n(w_1,\dots,w_n)\cdot|q(z)|$$
$$= \delta_n(K)^{\frac{n(n-1)}{2}}\cdot|q(z)|.$$

Thus,

$$\delta_n(K)^{\frac{n(n-1)}{2}}\,|q(z)| \leq \delta_{n+1}(K)^{\frac{n(n+1)}{2}} \overset{(i)}{\leq} \delta_n(K)^{\frac{n(n+1)}{2}}$$

$$\Longrightarrow \quad |q(z)| \leq \delta_n(K)^n.$$

Therefore $\|q\|_K^{1/n} \leq \delta_n(K)$, as desired.