UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 4 A
Some application of Exercise 2

Exercise 2 on Sheet 4 A provides the following permanence property of subharmoniticity.

Lemma 1. *Let T be a compact topological space and let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. Suppose that $f : \Omega \times T \to [-\infty, +\infty)$ is a function with the following properties:*

- *f is upper semicontinuous on $\Omega \times T$;*
- *the function*
$$f(\cdot, t) :\ \Omega \longrightarrow [-\infty, +\infty), \quad x \longmapsto f(x, t)$$
is subharmonic on Ω for each $t \in T$.

Then, we obtain a well-defined subharmonic function $s : \Omega \to [-\infty, +\infty)$ by
$$s(x) := \sup_{t \in T} f(x, t) \qquad \text{for } x \in \Omega.$$

This criterion can be used to verify the following assertion which we have made in Example 5.6 (ii) of the lecture.

Lemma 2. *For every open subset $\emptyset \neq \Omega \subsetneq \mathbb{C}$, the function*
$$s :\ \Omega \longrightarrow \mathbb{R}, \quad z \longmapsto -\log(\operatorname{dist}(z, \partial\Omega))$$
is subharmonic.

Proof. The crucial observation which allows us to apply the criterion provided by Lemma 1 is that $\operatorname{dist}(z, \partial\Omega) = \inf_{\zeta \in \partial\Omega} |z - \zeta|$ holds for every $z \in \Omega$; hence, as the function $x \mapsto -\log(x)$ is strictly decreasing on $(0, \infty)$, we conclude that $s(z) = \sup_{\zeta \in \partial\Omega} f(z, t)$ holds for all $z \in \Omega$, where $f : \Omega \times \partial\Omega \to \mathbb{R}$ is defined by $f(z, \zeta) := -\log|z - \zeta|$. The function f is continuous on $\Omega \times \partial\Omega$ (in particular, upper semicontinuous) and $f(\cdot, \zeta)$ is harmonic (in particular, subharmonic) for every fixed $\zeta \in \partial\Omega$ because it is locally given as the real part of a holomorphic function (see Exercise 1, Assignment 1 B).
At this stage, however, we can apply Lemma 1 only if $\partial\Omega$ is compact, as this is required by the criterion. In order to prove the assertion in full generality, we need to modify the

argument above. First of all, we notice that being subharmonic is a local property (see Theorem 5.8); thus, it suffices to check that $s|_{D(z_0,r)}$ is subharmonic for every disc $D(z_0, r)$ satisfying $\overline{D(z_0, r)} \subset \Omega$. The idea is that this local version should allow us to neglect the parts of $\partial\Omega$ which are sufficiently distant from $D(z_0, r)$; in other words, we want to find a compact subset T of $\partial\Omega$ such that $\text{dist}(z, \partial\Omega) = \text{dist}(z, T)$ holds for every $z \in D(z_0, r)$. Once we have found such a set T, we can consider the function $f : D(z_0, r)\times T \to \mathbb{R}$ defined like above by $f(z, \zeta) := -\log|z-\zeta|$, which then satisfies all conditions of Lemma 1; thus, we deduce that $z \mapsto \sup_{\zeta\in T} f(z, \zeta)$ is subharmonic on $D(z_0.r)$. Since $\sup_{\zeta\in T} f(z, \zeta) = -\log(\text{dist}(z, T)) = s(z)$ for all $z \in D(z_0, r)$, we conclude that $s|_{D(z_0,r)}$ is subharmonic, which proves the assertion.

Thus, it remains to construct the compact set T. For that purpose, we put $R_0 := \text{dist}(z_0, \partial\Omega) > 0$ and choose any $R > R_0 + 2r$. We claim that $T := \overline{D(z_0, R)} \cap \partial\Omega$ does the job.

This can be verified in two steps. First, we notice that

$$\text{dist}(z, \partial\Omega) \leq R_0 + r \qquad \text{for all } z \in D(z_0, r). \tag{1}$$

Indeed, by the triangle inequality, we see that for all $z \in D(z_0, r)$ and for every $\zeta \in \partial\Omega$

$$\text{dist}(z, \partial\Omega) \leq |\zeta - z| \leq |\zeta - z_0| + |z - z_0| < |\zeta - z_0| + r.$$

Hence, by passing to the infimum over all $\zeta \in \partial\Omega$, we infer from the latter that $\text{dist}(z, \partial\Omega) \leq \text{dist}(z_0, \partial\Omega) + r \leq R_0 + r$, which is the bound stated in (1).

With the help of (1), we can prove that $\text{dist}(z, \partial\Omega) = \text{dist}(z, T)$ holds for every $z \in D(z_0, r)$. Let us fix $z \in D(z_0, r)$. Since $T \subseteq \partial\Omega$, we obviously have that $\text{dist}(z, \partial\Omega) \leq \text{dist}(z, T)$. In order to prove the opposite inequality, we proceed as follows. Take any $\zeta \in \partial\Omega$ which satisfies $|z - \zeta| = \text{dist}(z, \partial\Omega)$. The triangle inequality yields then that $|\zeta - z_0| \leq |\zeta - z| + |z - z_0| \leq \text{dist}(z, \partial\Omega) + r$. Using (1), we conclude that $|\zeta - z_0| \leq R_0 + 2r < R$, i.e., $\zeta \in T$. Thus, $\text{dist}(z, T) \leq |\zeta - z| = \text{dist}(z, \partial\Omega)$, as desired. □