UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Assignments for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Assignment 4 B
for the tutorial on *Monday, July 6, 1:00 pm*

Exercise 1. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and bounded with piecewise smooth boundary $\partial\Omega$. Suppose further that $\Omega_0 \subseteq \mathbb{R}^N$ is an open subset such that $\overline{\Omega} \subset \Omega_0$. Deduce from Gauss' divergence theorem (Theorem 3.4 in the lecture) that *Green's identity*

$$\int_\Omega \big(u(x)\Delta v(x) - v(x)\Delta u(x)\big)\, \mathrm{d}\lambda^N(x) = \int_{\partial\Omega} \big(u(x)D_n v(x) - v(x)D_n u(x)\big)\, \mathrm{d}\sigma_{\partial\Omega}(x)$$

holds for all $u, v \in C^2(\Omega_0)$, where $\sigma_{\partial\Omega}$ is the surface measure on $\partial\Omega$, $n : \partial\Omega \to \mathbb{R}^N$ are the outer unit normal vectors to the surface $\partial\Omega$, and D_n denotes the directional derivative in the direction n, i.e., $D_n u(x) := \langle \operatorname{grad} u(x), n(x)\rangle$.

Hint: Apply Gauss' divergence theorem to both $u \cdot \operatorname{grad} v$ and $v \cdot \operatorname{grad} u$.

Exercise 2. Let $\emptyset \neq K \subset \mathbb{C}$ be compact. For any given $2 \leq n \in \mathbb{N}$, we call

$$\delta_n(K) := \max_{(w_1,\dots,w_n)\in K^n} \prod_{1\leq i<j\leq n} |w_i - w_j|^{\frac{2}{n(n-1)}}$$

the *n-th diameter of K*; an n-tuple $(w_1, \dots, w_n) \in K^n$ for which the maximum is attained is called a *Fekete n-tuple for K*. A *Fekete polynomial for K of degree n*, for $2 \leq n \in \mathbb{N}$, is a polynomial q of the form

$$q(z) = \prod_{j=1}^{n} (z - w_j)$$

where $(w_1, \dots, w_n) \in K^n$ is a Fekete n-tuple for K. Prove the following assertions:

(i) The sequence $(\delta_n(K))_{n=2}^\infty$ is decreasing.

(ii) If q is any Fekete polynomial for K of degree $n \geq 2$, then

$$\|q\|_K^{1/n} \leq \delta_n(K) \qquad \text{where} \qquad \|q\|_K := \max_{z\in K} |q(z)|.$$