Potential Theory in the Complex Plane

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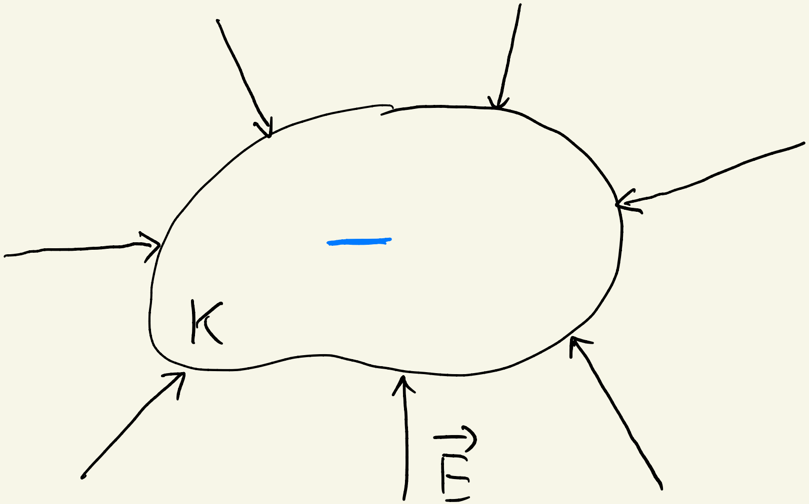
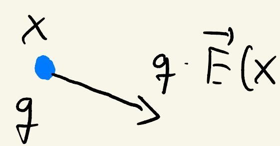
1. A physical motivation of potential theory

Potential theory has its origins in mathematical physics of the 19th century, namely in the study of gravity and the electrostatic force. Let us take a look at electrostatics.

Consider a (negatively) charged body $K \subset \mathbb{R}^3$. The body is surrounded by an electric field $\vec{E}$, i.e., the force acting on a test particle with the charge

q at the position $x = (x_1, x_2, x_3)$ is $q \cdot \vec{E}(x)$.





$(q < 0)$

By Coulomb's law, a particle with the charge q_0 at the point $x^0 = (x_1^0, x_2^0, x_3^0)$ induces the electric field

$$\vec{E}(x) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0}{\|x - x^0\|^3}(x - x^0),$$

where ε_0 is the vacuum permittivity and

$$\|x - x^0\| := \left(|x_1 - x_1^0|^2 + |x_2 - x_2^0|^2 + |x_3 - x_3^0|^2 \right)^{1/2}$$

At the end of the 18th century, it was observed by Lagrange that there exists a scalar-valued function Φ, called the potential of $\vec{E}$, such that

$$\vec{E} = -\operatorname{grad} \Phi,$$

when $\operatorname{grad} \Phi := \left(\frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2}, \frac{\partial \Phi}{\partial x_3} \right)$; indeed,

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0}{\|x - x^0\|}.$$

This has the advantage that the work done by the electric field $\vec{E}$ when it moves a particle with charge q from a point x^1 to a point x^2 along the path $\gamma : [t_1, t_2] \to \mathbb{R}^3$, $\gamma(t_1) = x^1$, $\gamma(t_2) = x^2$,

$$W = \int_\gamma q \cdot \vec{E}(x) \cdot dx = q \cdot \int_{t_1}^{t_2} < \vec{E}(\gamma(t)), \gamma'(t) > dt$$

can be computed easily: indeed,

$$< \vec{E}(\gamma(t)), \gamma'(t) > = - < \operatorname{grad} \Phi(\gamma(t)), \gamma'(t) >$$

$$= -\left(\frac{\partial \Phi}{\partial x_1}(\gamma(t)) \cdot \gamma_1'(t) + \frac{\partial \Phi}{\partial x_2}(\gamma(t)) \cdot \gamma_2'(t) + \frac{\partial \Phi}{\partial x_3}(\gamma(t)) \gamma_3'(t) \right)$$

$$= -(\Phi \circ \gamma)'(t)$$

so that $\quad W = -q \cdot \left(\Phi(x^2) - \Phi(x^1) \right).$

Now, suppose that we have charges which are "continuously" distributed over the body K, i.e., there is a function $\rho : K \to \mathbb{R}$ such that

$$\int_B \rho(x^0)\, dx_1^0\, dx_2^0\, dx_3^0$$

yields the charge of the portion $B \subseteq K$; we call ρ the charge density. Then, the electric field surrounding the body K is given by ($x \in \Omega$, $\Omega := \mathbb{R}^3 \setminus K$)

$$\vec{E}(x) = \frac{1}{4\pi\varepsilon_0}\int_K \frac{\rho(x^0)}{\|x-x^0\|^3}(x-x^0)\,dx_1^0\,dx_2^0\,dx_3^0$$

which has the potential

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_K \frac{\rho(x^0)}{\|x-x^0\|}\,dx_1^0\,dx_2^0\,dx_3^0 .$$

One can check that $\Phi : \Omega \to \mathbb{R}$ satisfies

$$\Delta \Phi \equiv 0,$$

where $\Delta := \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2$ is the Laplace operator, i.e., Φ is a harmonic function.

Potential theory explains, in particular, how ρ can be recovered from (a suitable extension of) Φ; this leads to the differential form of Gauss's law.

2. Harmonic functions

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be an open subset. Note that we suppose that $\mathbb{R}^N$ is endowed with the euclidean norm

$$\|x\| := \left(\sum_{i=1}^{N} |x_i|^2\right)^{1/2} \quad \text{for } x = (x_1, \dots, x_N) \in \mathbb{R}^N$$

and the corresponding inner product

$$\langle x, y \rangle := \sum_{i=1}^{N} x_i y_i \quad \text{for } \begin{array}{l} x = (x_1, \dots, x_N) \\ y = (y_1, \dots, y_N) \end{array} \in \mathbb{R}^N.$$

We denote the space of all functions $f : \Omega \to \mathbb{R}$

which are

- continuous by $C(\Omega) = C^0(\Omega)$,
- k times continuously differentiable by $C^k(\Omega)$,
- smooth, i.e., arbitrarily often cont. differentiable by $C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega)$.

2.1. Def.:

A function $f : \Omega \to \mathbb{R}$ is called harmonic (on Ω) if $f \in C^2(\Omega)$ and if f solves Laplace's equation

$$\frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_N^2} \equiv 0$$

(i.e., $\Delta f \equiv 0$, where $\Delta := \left(\frac{\partial}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial}{\partial x_N}\right)^2 : C^2(\Omega) \to C(\Omega)$ denote the Laplace operator).

We denote by $H(\Omega)$ the set of all harmonic functions.

2.2. Remark:

(i) Since $\Delta : C^2(\Omega) \to C(\Omega)$ is linear and

$$H(\Omega) = \ker\left(\Delta : C^2(\Omega) \to C(\Omega)\right),$$

it follows that $H(\Omega)$ is a vector space. It contains all (locally) affine linear functions (i.e., $f(x) = \langle x, a \rangle + b$ with $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$).

For $N = 1$, these are clearly all harmonic functions; thus, $N \geq 2$ in the following.

(ii) If $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry (i.e., $\phi(x) = a + Qx$ for an orthogonal matrix $Q \in M_N(\mathbb{R})$ and $a \in \mathbb{R}^N$)

or a dilation (i.e., $\phi(X) = \alpha X$ for some real number $\alpha > 0$), then $f \circ \phi \in H(\Omega)$ for all $f \in H(\phi(\Omega))$.

(iii) If $\Omega' \subseteq \Omega$ is an open, non-empty subset, then clearly $f|_{\Omega'} \in H(\Omega')$ for all $f \in H(\Omega)$.

2.3. Theorem:

Let $Y \in \mathbb{R}^N$ be given. Then $U_Y : \mathbb{R}^N \setminus \{Y\} \to \mathbb{R}$ defined by $(X \in \mathbb{R}^N \setminus \{Y\})$

$$U_y(x) = \begin{cases} -\log \|x-y\| \ , & \text{if } N=2 \\ \|x-y\|^{2-N} \ , & \text{if } N \geq 3 \end{cases}$$

is harmonic on $\mathbb{R}^N \setminus \{y\}$. Moreover, if f is harmonic on some annular region

$$A(y; r_1, r_2) := \{x \in \mathbb{R}^N \mid r_1 < \|x-y\| < r_2\}$$

with $0 \leq r_1 < r_2 \leq \infty$ and depends only on $\|x-y\|$, then there are $\alpha, \beta \in \mathbb{R}$ such that $f = \alpha U_y + \beta$

Proof: Suppose that $f \in C^2(A(y; r_1, r_2))$ depends only on $\|x - y\|$, i.e., there exists a function $F \in C^2((r_1, r_2))$ such that $f(x) = F(\|x - y\|)$ for all $x \in A(y; r_1, r_2)$. Put $r := \|x - y\|$. Then

$$\frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r}, \quad \text{so that} \quad \frac{\partial f}{\partial x_i}(x) = F'(r) \cdot \frac{x_i - y_i}{r}$$

for $i = 1, \ldots, N$, an

$$\frac{\partial^2 f}{\partial x_i^2}(x) = F''(r)\left(\frac{x_i - y_i}{r}\right)^2 + F'(r) \cdot \left(\frac{1}{r} - \frac{(x_i - y_i)^2}{r^3}\right)$$

Thus $\Delta f(x) = F''(r) + (N-1) \cdot \frac{1}{r} \cdot F'(r)$.

Hence, f is harmonic on $A(y; r_1, r_2)$ if and only if F solves the ordinary differential equation

$$F''(r) + (N-1) \cdot \frac{1}{r} \cdot F'(r) = 0 \qquad \forall r \in (r_1, r_2).$$

The only solutions are of the form $(r \in (r_1, r_2))$

$$F(r) = \begin{cases} -\alpha \log(r) + \beta & , \text{ if } N = 2 \\ \alpha r^{2-N} + \beta & , \text{ if } N \geq 3 \end{cases}.$$

So, both assertions of the theorem follow immediately $\square$

2.4. Definition:

We call the function u_y as defined in Theorem 2.3 the fundamental harmonic function for $\mathbb{R}^N$ with pole at y.

Potential theory is the study of harmonic functions in the same way as function theory is the study of holomorphic functions.

2.5. Remark:

$H(\Omega)$ is a vector space but <u>no</u> algebra (with respect to the pointwise multiplication). In fact, one has for $f, g \in H(\Omega)$ that

$$f \cdot g \in H(\Omega) \iff \langle \operatorname{grad} f, \operatorname{grad} g \rangle \equiv 0 \text{ on } \Omega.$$

(Recall that $\operatorname{grad} f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} \right)$ for every $f \in C^1(\Omega)$.)

In the case $N = 2$, one observes a close relationship between potential theory and function theory.

We identify $\mathbb{C} = \mathbb{R}^2$ in the usual way; $\mathbb{C} \ni z = x + iy$.

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. We say that $f : \Omega \to \mathbb{C}$ is **holomorphic on Ω** if, for all $z_0 \in \Omega$, the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists; we denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions $f : \Omega \to \mathbb{C}$. For $f \in \mathcal{O}(\Omega)$, in particular the partial derivatives do exist since for $z_0 = x_0 + i y_0$

$$(2.1) \quad \frac{\partial f}{\partial x}(z_0) = \lim_{x \to x_0} \frac{f(x_0 + i y_0) - f(x_0 + i y_0)}{x - x_0} = f'(z_0)$$

$$\frac{\partial f}{\partial y}(z_0) = \lim_{y \to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{y-y_0} = i f'(z_0)$$

For a partially differentiable function $f : \Omega \to \mathbb{C}$, we define the Pompeiu - Wirtinger derivatives by

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right) \quad \text{and}$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right)$$

For $f \in \mathcal{O}(\Omega)$, we see that

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

Function theory teaches us that $\mathcal{O}(\Omega) \subset C^\infty(\Omega, \mathbb{C})$ and that for a function $f : \Omega \to \mathbb{C}$

$$f \text{ holomorphic} \iff f \in C^1(\Omega, \mathbb{C}) \text{ and } \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \Omega.$$

For $\Delta : C^2(\Omega, \mathbb{C}) \to C(\Omega, \mathbb{C})$, $u + iv \mapsto \Delta u + i \Delta v$, we find that

$$(2.2) \quad \Delta f = 4 \frac{\partial^2 f}{\partial z \partial \bar{z}} = 4 \frac{\partial^2 f}{\partial \bar{z} \partial z} \quad \forall f \in C^2(\Omega, \mathbb{C}).$$

Hence, for every $f \in \mathcal{O}(\Omega)$, $\Delta f \equiv 0$ and thus

$$\mathrm{Re}(f), \mathrm{Im}(f) \in H(\Omega).$$

2.6. Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be a simply connected domain. For every $u \in H(\Omega)$, there exists a $f \in \mathcal{O}(\Omega)$ such that

$$u = \mathrm{Re}(f).$$

(We call $v := \operatorname{Im}(f)$ the harmonic conjugate of u.)

<u>Proof</u>:

Take $u \in \mathcal{H}(\Omega)$. Then $h := 2 \dfrac{\partial u}{\partial z} \in C^1(\Omega, \mathbb{C})$ and

$$0 \equiv \Delta u \overset{(2.2)}{=} 4 \frac{\partial^2 u}{\partial \bar{z} \partial z} = 2 \frac{\partial h}{\partial \bar{z}}.$$

Thus, $h \in \mathcal{O}(\Omega)$. Since Ω is simply connected, there exists a function $f_0 \in \mathcal{O}(\Omega)$ with $f_0' = h$. Because

$$\frac{\partial}{\partial x} (\operatorname{Re}(f_0) - u) = \operatorname{Re}\left(\frac{\partial f_0}{\partial x}\right) - \frac{\partial u}{\partial x} \overset{(2.1)}{=} \operatorname{Re}(f_0') - \operatorname{Re}(h) = 0,$$

$$\frac{\partial}{\partial y}(\mathrm{Re}(f_0) - u) = \mathrm{Re}\left(\frac{\partial f_0}{\partial y}\right) - \frac{\partial u}{\partial y} \overset{(2.1)}{=} -\mathrm{Im}(f_0') + \mathrm{Im}(h) = 0,$$

we conclude that $\mathrm{Re}(f_0) - u : \Omega \to \mathbb{R}$ is constant, say $\mathrm{Re}(f_0) - u \equiv C$ for some $C \in \mathbb{R}$. Then, $f := f_0 - C$ does the job.

$\square$

3. The mean value property and its consequences

3.1. Motivation:

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and consider $f \in \mathcal{O}(\Omega)$.
For $z_0 \in \Omega$ and $r > 0$ with $\overline{D(z_0, r)} \subset \Omega$, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_{z_0,r,\circlearrowleft}} \frac{f(z)}{z - z_0} \, dz,$$

where $\gamma := \gamma_{z_0,r,\circlearrowleft} : [0, 2\pi] \to \mathbb{C}$, $t \mapsto z_0 + re^{it}$. Then,

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)-z_0}\underbrace{\gamma'(t)}_{=ire^{it}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$$

$$= \frac{1}{2\pi}\int_{S^1} f(z_0+r\zeta)\,d\sigma^1(\zeta)\,,$$

$$\left(\sigma^1(\{e^{it} \mid t\in(t_1,t_2]\}) = t_2 - t_1\right)$$

i.e., f has the mean value property; in particular,

$$f(z_0)\cdot\underbrace{\int_0^{r_0} r\,dr}_{=\frac{1}{2}r_0^2} = \frac{1}{2\pi}\int_0^{r_0}\int_0^{2\pi} f(z_0+re^{it})\,r\,dt\,dr$$

and hence $\quad f(z_0) = \dfrac{1}{\pi r_0^2} \displaystyle\int_{D(z_0, r_0)} f(z) \, d\lambda^2(z).$

This has many important consequences such as the maximum modulus principle. From Theorem 2.6, it follows that every $u \in H(\Omega)$ has the mean value property. This is true not only for $N = 2$ but in full generality!

3.2. Definition:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $u \in C(\Omega)$.

We say that "u has the mean value property on Ω" (MVP) if for each $x_0 \in \Omega$ and every $r > 0$ with $\overline{B(x_0, r)} \subset \Omega$, where $B(x_0, r) := \{x \in \mathbb{R}^N \mid \|x - x_0\| < r\}$,

$$u(x_0) = \frac{1}{N\omega_N} \int_{\mathbb{S}^{N-1}} u(x_0 + r\varsigma)\, d\sigma^{N-1}(\varsigma) =: \mathcal{M}(u; x_0, r)$$

or equivalently (Exercise 1B-2(ii))

$$u(x_0) = \frac{1}{\omega_N r^N} \int_{B(x_0,r)} u(x)\, d\lambda^N(x) =: \mathcal{A}(u; x_0, r)$$

where • λ^N is the Lebesgue measure on $\mathbb{R}^N$

- σ^{N-1} is the spherical measure on $\mathbb{S}^{N-1}$,

$\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ defined by

$$\sigma^{N-1}(A) := N\lambda^N(\{ta \mid a \in A,\ t \in [0,1]\});$$

$$(3.1)\quad \int_{B(x_0,r_0)} f(x)\, d\lambda^N(x) = \int_0^{r_0} r^{N-1} \int_{\mathbb{S}^{N-1}} f(x_0 + r\xi)\, d\sigma^{N-1}(\xi)\, dr$$

- $\omega_N := \lambda^N(B(0,1)) = \dfrac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}$,

with the gamma function Γ; note that

$$\lambda^N(B(x_0,r)) = \omega_N r^N \quad \text{and} \quad \sigma^{N-1}(\mathbb{S}^{N-1}) = N\omega_N .$$

We want to prove:

3.3. Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. Then every $u \in H(\Omega)$ has the MVP.

The proof relies on the following fact

3.4 Theorem: (Gauss' Divergence Theorem)

If $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ is open and has a piecewise smooth boundary $\partial\Omega$, then every $F = (F_1, \dots, F_N) \in \left(C(\overline{\Omega}) \cap C^1(\Omega)\right)^N$ satisfies

$$\int_{\Omega} (\operatorname{div} F)(x)\, d\lambda^N(x) = \int_{\partial\Omega} \langle F(x), n(x) \rangle \, d\sigma_{\partial\Omega}(x),$$

where
- $n : \partial\Omega \to \mathbb{R}^N$ are the outer unit normal vectors to the surface $\partial\Omega$
- $\sigma_{\partial\Omega}$ is the surface measure on $\partial\Omega$, and
- $\operatorname{div} F$ is the divergence of F, which is

defined by $\operatorname{div} F(x) = \sum_{i=1}^{N} \frac{\partial F_i}{\partial x_i}(x)$.

<u>Proof of Theorem 3.3</u>:

We apply Theorem 3.4 to $B(x_0,r)$ and $\operatorname{grad} u$; since $n(x) = \frac{1}{r}(x - x_0)$, we get for $u \in C^2(\Omega)$

$$\int_{B(x_0,r)} \underbrace{(\operatorname{div} \operatorname{grad} u)}_{= \Delta u}(x) \, d\lambda^N(x)$$

$$= \int_{\partial B(x_0,r)} \langle \operatorname{grad} u(x), \frac{1}{r}(x - x_0) \rangle \, d\sigma_{\partial B(x_0,r)}(x)$$

and hence (note

$$\int_{\partial B(x_0,r)} f(x)\, d\sigma_{\partial B(x_0,r)}(x) = r^{N-1} \int_{S^{N-1}} f(x_0 + r\varsigma)\, d\sigma^{N-1}(\varsigma) \quad)$$

we get

$$\int_{B(x_0,r)} (\Delta u)(x)\, d\lambda^N(x) = r^{N-1} \int_{S^{N-1}} \underbrace{\langle \operatorname{grad} u(x_0 + r\varsigma), \varsigma \rangle}_{= \frac{\partial}{\partial r} u(x_0 + r\varsigma)} d\sigma^{N-1}(\varsigma)$$

$$= r^{N-1} \frac{d}{dr} \int_{S^{N-1}} u(x_0 + r\varsigma)\, d\sigma^{N-1}(\varsigma),$$

which yields:

$$(3.2) \quad r\,\mathcal{A}(\Delta u; x_0, r) = N \frac{d}{dr} \mathcal{M}(u; x_0, r)$$

Thus, if $\Delta u \equiv 0$, then $\mathcal{M}(u; x_0, \cdot) : (0, r_0) \to \mathbb{R}$ must be constant, where $r_0 > 0$ is such that $B(x_0, r_0) \subseteq \Omega$. By Exercise 1B-2(i), $\lim_{r \downarrow 0} \mathcal{M}(u; x_0, r) = u(x_0)$; thus

$$\mathcal{M}(u; x_0, r) = u(x_0) \qquad \forall\, r \in (0, r_0)$$

$\square$

We will see that the (local) MVP characterizes harmonic functions among continuous functions. The following result is a first step.

3.5. Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. Suppose that $u \in C(\Omega)$ has the MVP. Then $u \in C^\infty(\Omega)$.

<u>Proof:</u>

We consider $\phi \in C^\infty(\mathbb{R})$ such that $\phi(t) = 0$ for all $t \leq 0$ and

$$N \omega_N \int_0^1 t^{N-1} \phi(1-t^2)\, dt = 1.$$

For each $n \in \mathbb{N}$, we define $\phi_n \in C^\infty(\mathbb{R}^N)$ by

$$\phi_n(x) := n^N \phi(1 - n^2 \|x\|^2) \qquad \forall x \in \mathbb{R}^N.$$

Further, we put

$$\Omega_n := \begin{cases} \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\} & \text{if } \Omega \neq \mathbb{R}^N \\ \mathbb{R}^N & \text{if } \Omega = \mathbb{R}^N \end{cases}.$$

Since ϕ_n and all its derivatives are supported on $\overline{B(0, \frac{1}{n})}$, we see that

$$u_n(x) := \int_\Omega \phi_n(x - y)\, u(y)\, d\mathcal{L}^N(y), \qquad x \in \Omega_n$$

defines a smooth function $u_n : \Omega_n \to \mathbb{R}$. Now, for $x \in \Omega_n$,

$$u_n(x) = \int_{B(x,\frac{1}{n})} \underbrace{\phi_n(x-y)}_{=\phi_n(y-x)} u(y)\, d\lambda^N(y)$$

$$\overset{(3.1)}{=} \int_0^{1/n} r^{N-1} \int_{S^{N-1}} \underbrace{\phi_n(r\varsigma)}_{= n^N \phi(1-n^2r^2)} u(x+r\varsigma)\, d\sigma^{N-1}(\varsigma)\, dr$$

$$= \int_0^{1/n} r^{N-1} n^N \phi(1-n^2r^2) \underbrace{\int_{S^{N-1}} u(x+r\varsigma)\, d\sigma^{N-1}(\varsigma)}_{= N\omega_N M(u;x,r) = N\omega_N u(x)}\, dr$$

$$= u(x) \underbrace{N\omega_N \int_0^{1/n} r^{N-1} n^N \phi(1-n^2r^2)\, dr}_{=1} = u(x)$$

This tells us that $u|_{\Omega_n} = u_n$ is smooth. Since $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, it follows that $u \in C^\infty(\Omega)$.

$\square$

3.6. Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $u \in C(\Omega)$. TFAE

(i) $u \in H(\Omega)$

(ii) u has the MVP on Ω.

(iii) $\forall x_0 \in \Omega \ \exists r_0 > 0$: $B(x_0, r_0) \subseteq \Omega$ and $u|_{B(x_0,r)}$

has the MVP on $B(x_0, r_0)$.

Proof: (i) $\Rightarrow$ (ii): Theorem 3.3

(ii) $\Rightarrow$ (iii): trivial

(iii) $\Rightarrow$ (i): Due to Theorem 3.5, we have that $u|_{B(x_0,r_0)}$ is smooth. It suffices to show that $\Delta u(x_0) = 0$.

From (3.2) and Exercise 1B-2(i), we infer that

$$N\left(\mathcal{M}(u; x_0, r) - u(x_0)\right) = \int_0^r \rho\, \mathcal{A}(\Delta u; x_0, \rho)\, d\rho$$

for every $r \in (0, r_0)$, and thus, with Exercise 1B-2(i)

$$N \lim_{r \downarrow 0} \frac{1}{r^2} \big(\underbrace{\mathcal{M}(u; x_0, r) - u(x_0)}_{=0} \big) = \lim_{r \downarrow 0} \frac{1}{r^2} \int_0^r \rho \, A(\Delta u; x_0, \rho) \, d\rho$$

$$= \frac{1}{2} \Delta u(x_0)$$

The MVP enforces that $\Delta u(x_0) = 0$.

$\square$

3.7. Corollary:

For every open subset $\emptyset \neq \Omega \subseteq \mathbb{R}^N$, we have that

$$H(\Omega) \subset C^\infty(\Omega).$$

Moreover, if $u \in H(\Omega)$, then all partial derivatives of u belong to $H(\Omega)$.

Proof:

Take any $u \in H(\Omega)$. By Theorem 3.3, u has the MVP on Ω; thus, Theorem 3.5 yields that $u \in C^\infty(\Omega)$. The additional assertion follows by induction by using that $\Delta\left(\frac{\partial u}{\partial x_k}\right) = \frac{\partial}{\partial x_k}(\Delta u)$ for all $u \in C^\infty(\Omega)$.

$\square$

The following result is a "harmonic counterpart" of the maximum modulus principle for holomorphic functions.

3.8. Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $u \in \mathcal{H}(\Omega)$.

(i) If u attains a local maximum at some point $x_0 \in \Omega$ (i.e., $\exists r_0 > 0 : B(x_0, r_0) \subseteq \Omega$, $\forall x \in B(x_0, r_0)$: $u(x) \leq u(x_0)$), then u is constant in a neighborhood of x_0 (in fact, on $B(x_0, r_0)$).

The same conclusion holds for local minima.

(ii) If Ω is connected and u attains a local extremum at some point $x_0 \in \Omega$, then u is constant on Ω.

(iii) Let $\partial^\infty \Omega$ be the boundary of Ω in the one-point compactification $\mathbb{R}^N \cup \{\infty\}$ of $\mathbb{R}^N$; note that $\infty \in \partial^\infty \Omega$ if and only if Ω is unbounded.

If $u \in C(\Omega \cup \partial^\infty \Omega)$ is harmonic on Ω, then

$$(3.3) \quad \min_{x \in \partial^\infty \Omega} u(x) = \min_{x \in \Omega \cup \partial^\infty \Omega} u(x) \quad \text{and}$$

$$(3.4) \quad \max_{x \in \partial^\infty \Omega} u(x) = \max_{x \in \Omega \cup \partial^\infty \Omega} u(x).$$

Proof:

(i) Since u has the MVP, we see that for all $0 < r < r_0$

$$u(x_0) = A(u; x_0, r) = \frac{1}{\omega_N r^N} \int_{B(x_0,r)} \overbrace{u(x)}^{\leq u(x_0)} \, d\lambda^N(x) \overset{(*)}{\leq} u(x_0),$$

where, by continuity of u, $(*)$ would be strict if

$\{x \in B(x_0, r) \mid u(x) = u(x_0)\} \subsetneq B(x_0, r)$. Thus, u is constant on $B(x_0, r)$ for each $0 < r < r_0$ and so on $B(x_0, r_0)$.

(ii) Consider the set $\Omega_0 := \{ x \in \Omega \mid u(x) = u(x_0) \}$

Note that $\Omega_0 \neq \emptyset$ as $x_0 \in \Omega_0$.

By continuity of u, Ω_0 is closed relative to Ω.

Due to (i), Ω_0 is also open. Hence, as Ω is connected, it follows that $\Omega_0 = \Omega$, i.e., u is constant on Ω.

(iii) We prove (3.4); the proof of (3.3) is analogous.

It suffices to show "$\geq$"; "$\leq$" is obvious.

Take $x_0 \in \Omega \cup \partial^\infty \Omega$ such that

$$u(x_0) = \max_{x \in \Omega \cup \partial^\infty \Omega} u(x)$$

and let Ω_0 be any connected component of Ω for which $x_0 \in \Omega_0 \cup \partial^\infty \Omega_0$.

<u>Case 1</u>: $x_0 \in \Omega_0$

By (ii), it follows that u is constant on Ω_0 and so, by continuity of u, also on $\Omega_0 \cup \partial^\infty \Omega_0$

$$\Rightarrow \max_{x \in \partial^\infty \Omega} u(x) \geq \max_{x \in \partial^\infty \Omega_0} u(x) = u(x_0) = \max_{x \in \Omega \cup \partial^\infty \Omega} u(x)$$

<u>Case 2:</u> $x_0 \in \partial^\infty \Omega_0$

Then: $$\max_{x \in \partial^\infty \Omega} u(x) \geq \max_{x \in \partial^\infty \Omega_0} u(x) \geq u(x_0)$$

$$= \max_{x \in \Omega \cup \partial^\infty \Omega} u(x).$$

$\square$

3.9. Remark:

Note that the proof of Theorem 3.8 relies only on the MVP of u; due to Theorem 3.6, this is however equivalent to u being harmonic.

In fact, only the following condition is needed:

$$
\left.
\begin{aligned}
&\forall x_0 \in \Omega \quad \exists r_0 = r_0(x_0) > 0 : \quad B(x_0, r_0) \subseteq \Omega \quad \text{and} \\
&\qquad u(x) = \mathcal{A}(u; x_0, r) \qquad \forall\, 0 < r < r_0 \\
&\qquad \text{or equivalently} \\
&\qquad u(x) = \mathcal{M}(u; x_0, r) \qquad \forall\, 0 < r < r_0
\end{aligned}
\right\} \quad (3.5)
$$

This seems to be a weaker assumption, but in Chapter 4, we will prove the following result, which says that even (3.5) is equivalent to u being harmonic.

3.10 Theorem

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $u \in C(\Omega)$. TFAE

(i) $u \in H(\Omega)$.

(ii) u has the (local) MVP.

(iii) u satisfies condition (3.5).

The following is an analogue of Liouville's theorem for holomorphic functions.

3.11 Theorem:

Let $u \in H(\mathbb{R}^N)$ be bounded from below (or from above). Then u is constant.

<u>Proof:</u>

WLOG, we may suppose that $u(x) \geq 0$ for all $x \in \mathbb{R}^N$

Take $x, y \in \mathbb{R}^N$ and set $d := \|x - y\|$. Then, for every $r > 0$, $B(x,r) \subseteq B(y, r+d)$, and hence, by the MVP,

$$u(x) = A(u; x, r) \leq \frac{1}{\omega_N r^N} \int_{B(y, r+d)} u(x) \, d\lambda^N(x)$$

$$= \left(\frac{r+d}{r}\right)^N \mathcal{A}(u; y, r+d)$$

$$= \left(1 + \frac{d}{r}\right)^N u(y) \xrightarrow{r \nearrow \infty} u(y)$$

This shows that $u(x) \leq u(y)$. Since x, y were arbitrary, it follows that u is constant.

$\square$

4. The Poisson integral formula for a ball

The Poisson integral formula for balls can be seen as a "harmonic analogue" of Cauchy's integral formula for holomorphic functions on discs; see Exercise 2B-2.

4.1. Definition:

Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. The function

$$K_{x_0,r} : B(x_0,r) \times \partial B(x_0,r) \longrightarrow \mathbb{R},$$

$$K_{x_0,r}(x,y) := \frac{1}{N\omega_N r} \, \frac{r^2 - \|x-x_0\|^2}{\|x-y\|^N},$$

is called the Poisson kernel of $B(x_0,r)$.

4.2. Lemma:

Let $x_0 \in \mathbb{R}^N$ and $r>0$ be given. We have

$$K_{x_0,r}(\cdot,y) \in H(B(x_0,r))$$

for every fixed $y \in \partial B(x_0,r)$.

Proof: Exercise 2B-1. $\square$

4.3. Definition:

Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. For a signed measure $\mu : \mathcal{B}(\partial B(x_0, r)) \to \mathbb{R}$ (i.e., σ-additive, $\mu(\emptyset) = 0$; $\pm\infty$ are excluded) we call

$$I_{\mu, x_0, r} : B(x_0, r) \to \mathbb{R},$$

$$I_{\mu, x_0, r}(x) = \int_{\partial B(x_0, r)} K_{x_0, r}(x, y)\, d\mu(y),$$

the Poisson integral of μ.

Hahn - Jordan decomposition:

$\mu : \mathcal{B}(X) \to \mathbb{R}$ signed measure. Then:

$\exists \mu^{\pm}$ finite measures, $P, N \in \mathcal{B}(X)$ disjoint: $P \cup N = X$,

$\mu^+(N) = 0 = \mu^-(P)$, $\mu = \mu^+ - \mu^-$

We call $\|\mu\| := \mu^+(X) + \mu^-(X)$ the total variation of μ.

If $f : \partial B(x_0,r) \to \mathbb{R}$ is integrable w.r.t. the surface measure $\sigma_{\partial B(x_0,r)}$, then $I_{f,x_0,r} := I_{\mu,x_0,r}$ for the signed measure μ which is given by

$$d\mu(y) := f(y)\, d\sigma_{\partial B(x_0,r)}(y).$$

4.4. Theorem:

Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given.

(i) If μ is a signed measure on $\partial B(x_0,r)$, then

$$I_{\mu, x_0, r} \in H(B(x_0, r)).$$

(ii) If $f : \partial B(x_0, r) \to \mathbb{R}$ is integrable w.r.t. the surface measure $\sigma_{\partial B(x_0,r)}$, then, for $y \in \partial B(x_0, r)$,

$$\limsup_{B(x_0,r) \ni x \to y} I_{f, x_0, r}(x) \leq \limsup_{\partial B(x_0,r) \ni y \to x} f(z). \tag{4.1}$$

Further, if $f \in C(\partial B(x_0, r))$, then

$$\lim_{B(x_0,r) \ni x \to y} I_{f, x_0, r}(x) = f(y). \tag{4.2}$$

Proof:

(i) Take any $\overline{B(x,\rho)} \subset B(x_0,r)$. By Fubini's theorem, we get

$$A(I_{\mu,x_0,r};\, x,\rho) = \int_{\partial B(x_0,r)} \underbrace{A(K_{x_0,r}(\cdot,y);\, x,\rho)}_{=K_{x_0,r}(x,y) \text{ by Lemma 4.2 and Theorem 3.3}} \, d\mu(y)$$

$$= I_{\mu,x_0,r}(x)$$

Hence, $I_{\mu,x_0,r}$ has the MVP on $B(x_0,r)$; by

Theorem 3.6., we get $I_{\Gamma, x_0, r} \in H(B(x_0, r))$.

(ii) ① <u>Claim:</u> $I_{c, x_0, r} \equiv c$ for every $c \in \mathbb{R}$.

Note that $I_{c, x_0, r} \in H(B(x_0, r))$ and $I_{c, x_0, r}(x)$ depends only on $\|x - x_0\|$. Hence, Theorem 2.3 tells us that there are $\alpha, \beta \in \mathbb{R}$ such that

$$I_{c, x_0, r} = \alpha\, U_{x_0} + \beta \quad \forall x \in \underbrace{B(x_0, r) \setminus \{x_0\}}_{= A(x_0; 0, r)}.$$

Since $\lim_{x \to x_0} I_{c,x_0,r}(x) = I_{c,x_0,r}(x_0) = c$ (!), we must have $\alpha = 0$ and $\beta = c$, which yields that $I_{c,x_0,r} \equiv c$.

(2) Suppose that, for some $c \in \mathbb{R}$,

$$\limsup_{\partial B(x_0,r) \ni z \to y} f(z) < c.$$

(If no such c exists, i.e., if the limsup is ∞, then there is nothing to prove.)

Hence, there is some $\delta > 0$ such that

$$(4.3) \qquad \forall z \in B(y, 2\delta) \cap \partial B(x_0, r): \quad f(z) < c$$

Then, for every $x \in B(y, \delta) \cap B(x_0, r)$,

$$I_{f, x_0, r}(x) - c \overset{①}{=} I_{f-c, x_0, r}(x) = h_1(x) + h_2(x),$$

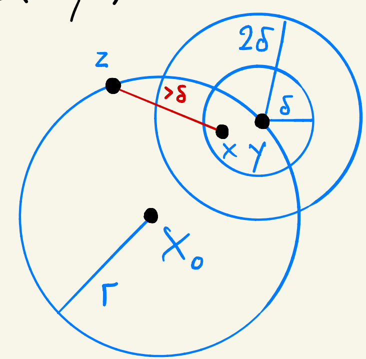
where

$$h_1(x) := \int_{\partial B(x_0, r) \setminus B(y, 2\delta)} K_{x_0, r}(x, z) \left(f(z) - c\right) d\sigma_{\partial B(x_0, r)}(z),$$

$$R_2(x) := \int_{\partial B(x_0,r)\cap B(y,2\delta)} K_{x_0,r}(x,z)\,(f(z)-c)\,d\sigma_{\partial B(x_0,r)}(z);$$

by (4.3), we have $R_2(x) < 0$, and further

$$|R_1(x)| \le \underbrace{\underbrace{\max_{z\in\partial B(x_0,r)\setminus B(y,2\delta)} K_{x_0,r}(x,z)}_{\le \frac{1}{N\omega_N r}\,\frac{r^2-\|x-x_0\|^2}{\delta^N}}}_{\longrightarrow\, 0 \text{ as } x\to y} \cdot \int_{\partial B(x_0,r)} (|f(z)|+|c|)\,d\sigma_{\partial B(x_0,r)}(z)$$



Hence, $\limsup\limits_{B(x_0,r)\ni y\to y} I_{f,x_0,r}(x) \le c$, which shows (4.1).

(iii) For $f \in C(\partial B(x_0,r))$, we apply (4.1) to f and $-f$, which leads to (4.2).

$\square$

4.4. Theorem: (Poisson's integral formula for balls)

Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. Then, every function $u \in C(\overline{B(x_0,r)}) \cap H(B(x_0,r))$ satisfies

$$u(x) = I_{u|_{\partial B(x_0,r)},x_0,r}(x) \qquad \forall x \in B(x_0,r)$$

Proof: By Theorem 4.4 (i), we have $v := u - I_{u,x_0,r} \in H(B(x_0,r))$, and due to Theorem 4.4 (ii), we have

$$\lim_{B(x_0,r) \ni x \to y} v(x) = 0 \qquad \forall y \in \partial B(x_0,r)$$

Thus, v extends to $v \in H(B(x_0,r)) \cap C(\overline{B(x_0,r)})$ with $v|_{\partial B(x_0,r)} \equiv 0$.

By Theorem 3.8 (iii), it follows $v \equiv 0$, as desired. $\square$

Proof of Theorem 3.10:

(i) $\Rightarrow$ (ii): Theorem 3.3. (ii) $\Rightarrow$ (iii): trivial

(iii) $\Rightarrow$ (i): Take any $x_0 \in \Omega$ and $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$. We consider $v := u - I_{u, x_0, r}$; by Theorem 4.4 (i), we have that $v \in C(B(x_0, r))$, and due to Theorem 4.4 (ii), we see that v extends to a function $v \in C(\overline{B(x_0, r)})$ with $v|_{\partial B(x_0, r)} \equiv 0$.

Since $I_{u, x_0, r}$ is harmonic on $B(x_0, r)$ and u satisfies (3.5) by assumption, also v satisfies (3.5).

Thus, Theorem 3.8 (iii) can be applied (see Remark 3.9) which gives $v \equiv 0$ on $\overline{B(x_0, r)}$ and so

$$u|_{B(x_0,r)} = I_{u,x_0,r} \in H(B(x_0,r))$$

Thus, in summary, $u \in H(\Omega)$, as desired. $\square$

4.6 Theorem: (Harnack's inequalities)

Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. Suppose that $u \in H(B(x_0,r))$ satisfies $u(x) \geq 0$ for all $x \in B(x_0,r)$. Then

$$\frac{(r - \|x-x_0\|)\, r^{N-2}}{(r + \|x-x_0\|)^{N-1}} \cdot u(x_0) \leq u(x) \leq \frac{(r + \|x-x_0\|)\, r^{N-2}}{(r - \|x-x_0\|)^{N-1}} u(x_0)$$

holds for every $x \in B(x_0,r)$.

Proof: Take any $0 < \rho < r$. Then, by Theorem 4.5,

$$u(x) = I_{u,x_0,\rho}(x) \qquad \forall\, x \in B(x_0, \rho).$$

Note that, for $x \in B(x_0, \rho)$ and $y \in \partial B(x_0, \rho)$,

$$\begin{aligned} K_{x_0,\rho}(x,y) &= \frac{1}{\omega_N N \rho} \cdot \frac{\rho^2 - \|x - x_0\|^2}{\|x - y\|^N} \\ &\geq \frac{1}{\omega_N N \rho} \cdot \frac{(\rho - \|x - x_0\|)(\rho + \|x - x_0\|)}{(\|y - x_0\| + \|x - x_0\|)^N} \\ &= \frac{1}{\omega_N N \rho} \cdot \frac{\rho - \|x - x_0\|}{(\rho + \|x - x_0\|)^{N-1}} \end{aligned}$$

and similarly,

$$K_{x_0,\rho}(x,y) \leq \frac{1}{\omega_N N \rho} \frac{\rho + \|x - x_0\|}{(\rho - \|x - x_0\|)^{N-1}}.$$

Hence, by the MVP of u,

$$u(x) = \int_{\partial B(x_0,\rho)} K_{x_0,\rho}(x,y)\, u(y)\, d\sigma_{\partial B(x_0,\rho)}(y)$$

$$\geq \frac{1}{\omega_N N \rho} \frac{\rho - \|x - x_0\|}{(\rho + \|x - x_0\|)^{N-1}} \underbrace{\int_{\partial B(x_0,\rho)} u(y)\, d\sigma_{\partial B(x_0,\rho)}(y)}_{= \rho^{N-1} N \omega_N\, u(x_0)}$$

$$= \frac{(\rho - \|x - x_0\|)\rho^{N-2}}{(\rho + \|x - x_0\|)^{N-1}} u(x_0),$$

and similarly

$$u(x) \leq \frac{(\rho + \|x - x_0\|)\rho^{N-2}}{(\rho - \|x - x_0\|)^{N-1}} u(x_0).$$

Letting $\rho \nearrow r$, we obtain the asserted bounds.

$\square$

4.7. Corollary:

In the situation of Theorem 4.6, it holds that

$$\| \operatorname{grad} u(x_0) \| \leq \frac{N}{r} u(x_0).$$

<u>Proof:</u> Let $e \in \mathbb{R}^N$ with $\|e\| = 1$ be given. Then

$$f : (-r, r) \to \mathbb{R}, \quad t \mapsto u(x_0 + te)$$

is well-defined and smooth with

$$f'(0) = \langle \operatorname{grad} u(x_0), e \rangle.$$

By Theorem 4.6, we further have for $t \in (0, r)$,

$$\frac{(r-t) r^{N-2}}{(r+t)^{N-1}} f(0) \leq f(t) \leq \frac{(r+t) r^{N-2}}{(r-t)^{N-1}} f(0).$$

Hence,

$$\underbrace{\frac{1}{t}\left(\frac{(r-t)r^{N-2}}{(r+t)^{N-1}}-1\right)f(0)}_{\to -\frac{N}{r}} \le \underbrace{\frac{f(t)-f(0)}{t}}_{\to f'(0)} \le \underbrace{\frac{1}{t}\left(\frac{(r+t)r^{N-2}}{(r-t)^{N-1}}-1\right)f(0)}_{\to \frac{N}{r}}.$$

if $t \to 0$.

Therefore, $\quad |\langle \operatorname{grad} u(x_0), e\rangle| = |f'(0)| \le \dfrac{N}{r}$,

from which the assertion follows.

$\square$

4.8. Remark:

The Dirichlet problem on $B(x_0,r)$ is to find, for a given $f \in C(\partial B(x_0,r))$, a function $u \in H(B(x_0,r))$ such that

$$\lim_{B(x_0,r) \ni x \to y} u(x) = f(y) \qquad \forall\, y \in \partial B(x_0,r).$$

Theorem 4.4 tells us that $u = I_{f,x_0,r}$ solves the Dirichlet problem. Due to Theorem 3.8, it is in fact the unique solution.

5. Subharmonic functions

Subharmonic functions generalize harmonic functions; they are more flexible but have similar strong properties. This class is crucial for the study of harmonic functions.

5.1 Definition:

Let X be a topological space. We say that a function $f : X \to [-\infty, +\infty)$ is

- upper semicontinuous (on X) if $f^{-1}([-\infty, a))$ is open in X for each $a \in \mathbb{R}$,

- lower semicontinuous (on X) if $-f$ is upper semicontinuous on X.

5.2. Definition:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. A function $s : \Omega \to [-\infty, +\infty)$ is called **subharmonic** (on Ω) if

(i) s is upper semicontinuous on Ω,

(ii) s has the **subharmonic MVP** (on Ω), i.e.

$$s(x) \leq \mathcal{M}(s; x, r) \text{ whenever } \overline{B(x,r)} \subset \Omega, \text{ and}$$

(iii) $s \not\equiv -\infty$ on each connected component of Ω.

We denote by $S(\Omega)$ the set of all subharmonic functions on Ω.

A function $u : \Omega \to (-\infty, +\infty]$ is called superharmonic (on Ω), if $-u : \Omega \to [-\infty, +\infty)$ is subharmonic; those functions have the superharmonic MVP (on Ω), i.e., $u(x) \geq \mathcal{M}(u; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$.

We denote by $\mathcal{U}(\Omega)$ the set of all superharmonic functions on Ω.

Note that: $\mathcal{H}(\Omega) = \mathcal{S}(\Omega) \cap \mathcal{U}(\Omega)$.

5.3. Remark:

(i) Let $f : X \to [-\infty, +\infty)$ be upper semicontinuous and let $K \subseteq X$ be compact, then $\sup_{x \in K} f(x) < \infty$ and there exists $x_0 \in K$ such that

$$f(x_0) = \sup_{x \in K} f(x).$$

(ii) Let $s : \Omega \to [-\infty, +\infty)$ be upper semicontinuous.

Define $s^+ : \Omega \to [0, +\infty)$ and $s^- : \Omega \to [0, +\infty]$ by

$$s^{\pm}(x) := \max\{\pm s(x), 0\} \quad \text{for } x \in \Omega;$$

then $s = s^+ - s^-$. Note that s^+ is upper semicontinuous.

Thus, whenever $\overline{B(x,r)} \subset \Omega$, then

$$\mathcal{M}(s; x, r) = \frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x,r)} s(\zeta)\, d\sigma(\zeta) \quad \left(\sigma := \sigma_{\partial B(x,r)}\right)$$

$$:= \underbrace{\frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x,r)} s^+(\zeta)\, d\sigma(\zeta)}_{\leq \max\limits_{\zeta \in \partial B(x,r)} s^+(\zeta) \overset{(i)}{<} \infty} - \underbrace{\frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x,r)} s^-(\zeta)\, d\sigma(\zeta)}_{\in [0, +\infty]}$$

Thus, $\mathcal{M}(s;x,r) \in [-\infty, \infty)$ is well-defined.

5.4 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and $s \in S(\Omega)$. Then:

(i) $\forall y \in \Omega$: $\limsup_{x \to y} s(x) = s(y)$.

(ii) $s(x) \leq \mathcal{A}(s;x,r)$ whenever $\overline{B(x,r)} \subset \Omega$.

(iii) s is locally integrable, i.e.

$$\int_K |s(x)| \, d\lambda^N(x) < \infty \qquad (5.1)$$

for every compact subset $K \subset \Omega$.

Proof:

(i) Since s is usc, we have

$$\limsup_{x \to y} s(x) \leq s(y).$$

If this inequality would be strict, then we could find $r_0 > 0$ such that $B(y, r_0) \subseteq \Omega$ and

$$\forall x \in B(y, r_0) \setminus \{y\} : \quad s(x) < s(y)$$

Then, for each $0 < r < r_0$, by Remark 5.3 (i)

$$\mathcal{M}(s; y, r) \leq \max_{\zeta \in \partial B(y,r)} s(\zeta) < s(y),$$

in contradiction to the subMVP.

(ii) Like in Ex 1B-2(ii), one finds that

$$r^N \mathcal{A}(s; x, r) = N \int_{[0,r)} \rho^N \mathcal{M}(s; x, \rho) \, d\lambda^1(\rho) \qquad (5.2)$$

and derives that $\mathcal{A}(s; x, r) \geq s(x)$.

(iii) By Heine-Borel, it suffices to show that for each $x \in \Omega$, one finds $r > 0$ such that $\overline{B(x,r)} \subset \Omega$ and (5.1) holds for $K = \overline{B(x,r)}$. In particular, without loss of generality, we may suppose that Ω is connected. Put

$$\Omega_0 := \left\{ x \in \Omega \,\middle|\, \exists r > 0 : \overline{B(x,r)} \subset \Omega, \int_{\overline{B(x,r)}} |s(x)| \, d\mathcal{L}^n(x) < \infty \right\}$$

① <u>Claim:</u> Ω_0 is open.

Let $x \in \Omega_0$ be given. Choose $r > 0$ such that $\overline{B(x,r)} \subset \Omega$

and $\int_{\overline{B(x,r)}} |s(x)|\, d\lambda^N(x) < \infty$. We want to show that $B(x,r) \subseteq \Omega_0$. Take $x' \in B(x,r)$ and set $r' := r - |x'-x| > 0$. Then, $B(x',r') \subseteq B(x,r)$ and

$$\int_{\overline{B(x',r')}} |s(x)|\, d\lambda^N(x) \leq \int_{\overline{B(x,r)}} |s(x)|\, d\lambda^N(x) < \infty,$$

so that $x' \in \Omega_0$.

② <u>Claim</u>: $\Omega \setminus \Omega_0$ is open and $s|_{\Omega\setminus\Omega_0} \equiv -\infty$

Let $x \in \Omega \setminus \Omega_0$ be given. Choose $r > 0$ such that

$\overline{B(x,2r)} \subseteq \Omega$. We want to show that $B(x,r) \subseteq \Omega \setminus \Omega_0$ and $s|_{B(x,r)} \equiv -\infty$. Take $x' \in B(x,r)$ and set $r' := r - |x'-x| \in (0,r]$. Then $\overline{B(x,r')} \subset \Omega$ and hence, as $x \in \Omega \setminus \Omega_0$, we must have

$$\int_{\overline{B(x,r')}} |s| \, d\lambda^N = \infty .$$

Since $\overline{B(x,r')} \subseteq \overline{B(x',r)}$, we infer that

$$\int_{\overline{B(x',r)}} |s| \, d\lambda^N = \infty \tag{5.3}$$

However, as $\overline{B(x',r)} \subseteq \overline{B(x,2r)} \subseteq \Omega$, we know that s is bounded from above on $\overline{B(x',r)}$; hence, (5.3) gives

$$\int_{\overline{B(x',r)}} s \, d\lambda^N = -\infty$$

and so $A(s;x',r) = -\infty$. Due to (ii), it follows that $s(x') = -\infty$. Therefore $s|_{B(x,r)} \equiv -\infty$ and consequently $B(x,r) \subseteq \Omega \setminus \Omega_0$.

③ By Definition 5.2 (iii), we know that $s \not\equiv -\infty$;
thus, $\Omega_0 \neq \emptyset$.

In summary, as Ω is connected, we get $\Omega = \Omega_0$.

$\square$

5.5 Corollary:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. For each $s \in \mathcal{S}(\Omega)$, we have

$$\lambda^N(\{x \in \Omega \mid s(x) = -\infty\}) = 0.$$

<u>Proof</u>: Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets $K_n \subset \Omega$

such that $\Omega = \bigcup_{n=1}^{\infty} K_n$. Put $E := \{x \in \Omega \mid s(x) = -\infty\}$.

Then $\lambda^N(K_n \cap E) = 0$ since $\int_{K_n} |s(x)| \, d\lambda^N(x) < \infty$ due to Theorem 5.4. Because $E = \bigcup_{n=1}^{\infty} (K_n \cap E)$, we get $\lambda^N(E) = 0$, as desired.

$\square$

5.6 Examples:

(i) If h is harmonic, then both $|h|$ and h^2 are subharmonic. This can be shown with the help of the triangle inequality and the Cauchy-Schwarz inequality, respectively.

(ii) It is a less obvious fact that

$$S : \Omega \to \mathbb{R}, \quad x \mapsto -\log\left(\operatorname{dist}(x, \partial\Omega)\right)$$

is subharmonic for every open set $\emptyset \neq \Omega \subsetneq \mathbb{C}$ with $\operatorname{dist}(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|$.

The maximum principle for harmonic functions (see Theorem 3.8) extends to the subharmonic case.

5.7 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $s \in S(\Omega)$.

(i) If s attains a local maximum at some point $x_0 \in \Omega$, then s is constant on some neighborhood of x_0.

(ii) If Ω is connected and s attains a local maximum at some point $x_0 \in \Omega$, then s is constant on Ω.

(iii) If $u \in U(\Omega)$ is such that

$$\limsup_{x \to y} \big(s(x) - u(x)\big) \leq 0 \quad \forall y \in \partial^\infty \Omega, \quad (5.4)$$

then $s(x) \leq u(x)$ for all $x \in \Omega$.

<u>Proof:</u>

(i) and (ii) can be shown like in the proof of Theorem 3.8. To prove (iii), we may suppose that

- $u \equiv 0$ (since $s - u \in S(\Omega)$),
- Ω is connected (since (5.4) holds for each connected component of Ω).

We define an usc function $\bar{s} : \Omega \cup \partial^\infty \Omega \to [-\infty, +\infty)$ by $\bar{s}|_\Omega = s$ and

$$\bar{s}(y) := \limsup_{x \to y} s(x) \qquad \forall y \in \partial^\infty \Omega$$

By Remark 5.3 (i), we find $x_0 \in \Omega \cup \partial^\infty \Omega$ such that

$$\bar{s}(x_0) = \sup_{x \in \Omega \cup \partial^\infty \Omega} s(x).$$

Assume that $\bar{s}(x_0) > 0$. Since $\bar{s} \leq 0$ on $\partial^\infty \Omega$ due to (5.4), it follows that $x_0 \in \Omega$. So, $\bar{s}$ and hence s attain a local (in fact, global) maximum at some point in Ω; by (ii), s and hence $\bar{s}$ have a positive constant value, in contradiction to (5.4). Hence, $\bar{s}(x_0) \leq 0$.

$\square$

Analogously, one has a minimum principle for superharmonic functions.

Our next goal is the following characterization of subharmonicity; see Theorems 3.6, 3.10, and 4.5.

5.8 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider a function $s : \Omega \to [-\infty, +\infty)$ which is usc and satisfies $s \not\equiv -\infty$ on each connected component of Ω. TFAE:

(i) $s \in S(\Omega)$

(ii) $s \leq I_{s,x,r}$ on $B(x,r)$ whenever $\overline{B(x,r)} \subset \Omega$

(iii) For each $x \in \Omega$ such that $s(x) > -\infty$, we have

$$\liminf_{r \downarrow 0} \frac{1}{r^2} \left(\mathcal{M}(s; x, r) - s(x) \right) \geq 0.$$

(iv) $\forall x \in \Omega \;\; \exists r_0 > 0 : \; B(x, r_0) \subseteq \Omega$ and

$$s(x) \le \mathcal{M}(s; x, r) \qquad \forall\, 0 < r < r_0 .$$

(v) $\forall x \in \Omega \;\; \exists r_0 > 0 : \; B(x, r_0) \subseteq \Omega$ and

$$s(x) \le \mathcal{A}(s; x, r) \qquad \forall\, 0 < r < r_0 .$$

(vi) If U is an open and bounded set with $\overline{U} \subset \Omega$ and if $h \in C(\overline{U}) \cap H(U)$ satisfies $s \le h$ on ∂U, then $s \le h$ on U.

The proof requires the following fact.

5.9. Lemma:

Let $\emptyset \neq X \subseteq \mathbb{R}^N$ be any subset and suppose that $f: X \to [-\infty, +\infty)$ is usc on X and bounded from above. Then there is a pointwise decreasing sequence $(f_n)_{n=1}^{\infty}$ in $C(\mathbb{R}^N)$ such that

$$f_n(x) \to f(x) \text{ as } n \to \infty \qquad \forall x \in X.$$

Proof: (sketch)

We extend f to an usc function

$$\bar{f} : \mathbb{R}^N \to [-\infty, +\infty), \quad x \longmapsto \begin{cases} f(x) & , \quad x \in X \\ \limsup\limits_{y \to x} f(y) & , \quad x \in \bar{X} \setminus X \\ -\infty & , \quad x \in \mathbb{R}^N \setminus \bar{X} \end{cases}$$

If $\bar{f} \equiv -\infty$ on $\mathbb{R}^N$, then $f_n \equiv -n$; otherwise, we define $f_n : \mathbb{R}^N \to \mathbb{R}$ by

$$f_n(x) := \sup_{y \in \mathbb{R}^N} \left(\bar{f}(y) - n\|x - y\| \right) \qquad \forall x \in \mathbb{R}^N$$

and check that $|f_n(x_1) - f_n(x_2)| \leq n \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^N$. Thus, in each case, $f_n \in C(\mathbb{R}^N)$ and one verifies that f_n is pointwise decreasing and convergent to $\overline{f}$.

$\square$

Proof of Theorem 5.8:

(i) $\Rightarrow$ (iv) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (vi): We consider $f: \mathbb{R}^N \to \mathbb{R}$, $y \mapsto \|y\|^2$ and put $a := \sup\{f(y) \mid y \in U\} < \infty$. For $\varepsilon > 0$, we put

$$u_\varepsilon := h - s - \varepsilon(f - a),$$

which is a lsc function on $\overline{U}$ satisfying $u_\varepsilon \geq 0$ on ∂U. We set

$$b_\varepsilon := \inf\{u_\varepsilon(y) \mid y \in \overline{U}\}.$$

1 <u>Claim</u>: $u_\varepsilon > b_\varepsilon$ on U

Like in the proof of Theorem 3.6, we infer from (3.2)

$$\lim_{r \downarrow 0} \frac{1}{r^2}\left(\mathcal{M}(f;y,r) - f(y)\right) = \frac{1}{2N}(\Delta f)(y) = 1$$

for all $y \in \mathbb{R}^N$. Thus, for every $y \in U$,

$$\mathcal{M}(u_\varepsilon; y, r) = \underbrace{\mathcal{M}(h; y, r)}_{= h(y)} - \mathcal{M}(s; y, r) - \varepsilon\left(\mathcal{M}(f; y, r) - a\right)$$

and hence, for all sufficiently small $r > 0$, by (iii)

$$\begin{aligned}\frac{1}{r^2}\left(\mathcal{M}(u_\varepsilon; y, r) - u_\varepsilon(y)\right) &= -\frac{1}{r^2}\left(\mathcal{M}(s; y, r) - s(y)\right) \\ &\quad - \varepsilon\frac{1}{r^2}\left(\mathcal{M}(f; y, r) - f(y)\right) < 0\end{aligned}$$

i.e., $\mathcal{M}(u_\varepsilon; y, r) < u_\varepsilon(y)$. Therefore, if there was a $y_0 \in U$ such that $u_\varepsilon(y_0) \leq b_\varepsilon$, then

$$b_\varepsilon \leq \inf \left\{ u_\varepsilon(y) \mid y \in B(y_0, r) \right\} \leq \mathcal{M}(u_\varepsilon; y_0, r) < u_\varepsilon(y) \leq b_\varepsilon$$

would yield a contradiction. Hence, $u_\varepsilon > b_\varepsilon$ on U.

2 Due to Remark 5.3 (i), we find $y_0 \in \overline{U}$ such that

$$u_\varepsilon(y_0) = b_\varepsilon .$$

By 1, we must have $y_0 \in \partial U$ and hence

$$b_\varepsilon = u_\varepsilon(y_0) \geq 0 .$$

Thus, $u_\varepsilon(y) \geq 0$ for all $y \in \overline{U}$.

③ Letting $\varepsilon \downarrow 0$, we infer from ② that $h \geq s$ on $\overline{U}$.

(vi) $\Rightarrow$ (ii): Consider $\overline{B(x,r)} \subset \Omega$. By Lemma 5.9, there exists a pointwise decreasing sequence $(f_n)_{n=1}^{\infty}$ in $C(\partial B(x,r))$ such that $f_n \to s$ on $\partial B(x,r)$. Define $h_n \in C(\overline{B(x,r)}) \cap H(B(x,r))$ by $h_n|_{B(x,r)} := I_{f_n,x,r}$ and $h_n|_{\partial B(x,r)} := f_n$; see Theorem 4.4(ii). By assumption (vi), $s \leq h_n$ on $B(x,r)$. The monotone convergence theorem implies that $h_n \to I_{s,x,r}$ on $B(x,r)$; therefore, $s \leq I_{s,x,r}$, as desired.

(ii) $\Rightarrow$ (i): If $\overline{B(x,r)} \subset \Omega$, then (ii) gives

$$s(x) \leq I_{s,x,r}(x) = \mathcal{M}(s;x,r),$$

which shows that s has the sub MVP on Ω, i.e., $s \in S(\Omega)$.

So far, we have shown the equivalence of (i), (ii), (iii), (iv), and (vi); we connect (v) as follows.

(i) $\Longrightarrow$ (v) : Theorem 5.4 (ii)

(v) $\Longrightarrow$ (iii) : For $0 < r < r_0$, we deduce from (5.2) that

$$0 \leq r^N \Big(\mathcal{A}(s;x,r) - s(x) \Big) = N \int_{[0,r)} \varrho^N \Big(\mathcal{M}(s;x,\varrho) - s(x) \Big) \, d\lambda^1(\varrho)$$

Thus, if we would have that

$$\limsup_{r \downarrow 0} \frac{1}{r^2} \big(\mathcal{M}(s;x,r) - s(x) \big) < 0,$$

then there would be some $a > 0$ such that

$$\frac{1}{r^2}\left(\mathcal{M}(s; x, r) - s(x)\right) \le -a < 0$$

for all sufficiently small values of r and hence

$$0 \le -Na \int_{[0,r)} \rho^{N+2} \, d\lambda^1(\rho) < 0,$$

a contradiction. Therefore, (iii) must hold.

$\square$

The characterization (vi) explains the name

subharmonic.

5.10 Corollary: (Weak identity principle)

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. If $s_1, s_2 \in S(\Omega)$ satisfy $s_1 = s_2$ λ^N-almost everywhere on Ω, i.e.,

$$\lambda^N\left(\{x \in \Omega \mid s_1(x) \neq s_2(x)\}\right) = 0,$$

then $s_1 \equiv s_2$ on Ω.

<u>Proof</u>:

① <u>Claim</u>: For $s \in S(\Omega)$ and every $x \in \Omega$,

$$\lim_{r \downarrow 0} \mathcal{A}(s; x, r) = s(x)$$

By Theorem 5.8 (v),

$$\liminf_{r \downarrow 0} \mathcal{A}(s; x, r) \geq s(x),$$

and by Theorem 5.4 (i),

$$\limsup_{r \downarrow 0} \mathcal{A}(s; x, r) \leq \limsup_{y \to x} s(y) = s(x)$$

Thus, $\lim_{r \downarrow 0} \mathcal{A}(s; x, r)$ exists and is $s(x)$.

(2) If $s_1 = s_2$ λ^N-almost everywhere, then, for any $x \in \Omega$, we have that

$$\mathcal{A}(s_1; x, r) = \mathcal{A}(s_2; x, r) \quad \forall\, 0 < r < r_0$$

whenever $B(x, r_0) \subset \Omega$; thus, by (1), $s_1(x) = s_2(x)$.

$\square$

5.11 Corollary:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $s \in C^2(\Omega)$. Then

$$s \in \mathcal{S}(\Omega) \iff \Delta s \geq 0 \text{ on } \Omega.$$

Proof:

Since for all $x \in \Omega$

$$\lim_{r \downarrow 0} \frac{1}{r^2}\Big(\mathcal{M}(s;x,r) - s(x)\Big) = \frac{1}{2N}(\Delta s)(x),$$

the asserted equivalence is (i) $\Longleftrightarrow$ (iii) in Theorem 5.8.

$\square$

In view of Corollary 5.11, it is desirable to know how to approximate subharmonic functions by smooth ones. For this purpose, we have the following.

5.12 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and consider $s \in S(\Omega)$. Further, let $\emptyset \neq U \subseteq \mathbb{R}^N$ be open and bounded with $\overline{U} \subset \Omega$. Then, there exists a sequence $(s_n)_{n=1}^{\infty}$ in $S(U) \cap C^{\infty}(U)$ which is pointwise decreasing on U with

$$s(x) = \lim_{n \to \infty} s_n(x) \qquad \forall x \in U.$$

We record an important consequence:

5.13 Theorem:

Let $\emptyset \neq \Omega_1, \Omega_2 \subseteq \mathbb{C}$ be domains and suppose that $f \in \mathcal{O}(\Omega_1)$ is non-constant with $f(\Omega_1) \subseteq \Omega_2$. Then:

$$s \in S(\Omega_2) \implies s \circ f \in S(\Omega_1).$$

<u>Proof</u>:

Take any $z_0 \in \Omega_1$. Choose $r_2 > 0$ such that

$U_2 := D(f(z_0), r_2)$ satisfies $\overline{U_2} \subset \Omega_2$; take any $r_1 > 0$ such that $U_1 := D(z_0, r_1)$ satisfies $U_1 \subseteq f^{-1}(U_2)$. It suffices to prove that $s \circ f|_{U_1} \in S(U_1)$ since being subharmonic is a local property due to Theorem 5.8 and since $z_0 \in \Omega_1$ was arbitrary.

By Theorem 5.12, we find a sequence $(s_n)_{n=1}^{\infty}$ in $S(U_2) \cap C^{\infty}(U_2)$ which is pointwise decreasing on U_2 with limit $s|_{U_2}$. Therefore, $(s_n \circ f|_{U_1})_{n=1}^{\infty}$ is pointwise decreasing on U_1 with limit $s \circ f|_{U_1}$; further,

$s_n \circ f|_{U_1} \in C^\infty(U_1)$, so that $s_n \circ f|_{U_1} \in S(U_1)$ as

$$\Delta(s_n \circ f)(z) \underset{\text{Ex 4A-1}}{=} \underbrace{(\Delta s_n)(f(z))}_{\geq 0} \, |f'(z)|^2 \geq 0 \qquad \forall z \in U_1$$

(see Corollary 5.11). By the open mapping theorem (FT, Satz 13.5), we know that $f(U_1)$ is open; thus, by Theorem 5.4 (iii), $s \not\equiv -\infty$ on $f(U_1)$ and so $s \circ f|_{U_1} \not\equiv -\infty$ on U_1. From Exercise 3B-2, it follows that $s \circ f|_{U_1} \in S(U_1)$, as desired.

$\square$

The proof of Theorem 5.12 uses convolution with smooth function like in Theorem 3.5. We skip the details. We note, however, that the proof uses the following fact, which is a consequence of Fatou's lemma and Fubini's theorem.

5.14 Theorem:

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and connected. Further, let (Y, μ) be a σ-finite measure space. Suppose that $f: \Omega \times Y \to [-\infty, +\infty]$ is measurable such that

- $\forall y \in Y: \quad f(\cdot, y) \in \mathcal{U}(\Omega)$
- $\exists g : Y \to \mathbb{R}$, μ-integrable:

$$\forall (x,y) \in \Omega \times Y: \quad f(x,y) \geq g(y)$$

Then

$$u(x) := \int_Y f(x,y)\, d\mu(y) \qquad \text{for } x \in \Omega$$

defines a function $u : \Omega \to [-\infty, +\infty]$ which satisfies either $u \equiv +\infty$ on Ω or $u \in \mathcal{U}(\Omega)$.

6. Riesz measure

In Corollary 5.11, we have seen that subharmonic C^2-function have a non-negative Laplacian. Here, we generalize this to arbitrary subharmonic functions: the Laplacian, if understood in a distributional sense, yields then the Riesz measure.

In the following, let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open.

We denote by

- $C_c(\Omega)$ the space of all continuous functions $f : \Omega \to \mathbb{R}$ having compact support

$$\operatorname{supp} f := \overline{\{x \in \Omega \mid f(x) \neq 0\}}^{\Omega}$$

- $C_c^\infty(\Omega)$ the space of all compactly supported smooth functions on Ω, i.e.,

$$C_c^\infty(\Omega) := C_c(\Omega) \cap C^\infty(\Omega).$$

6.1 Definition:

Let $u : \Omega \to [-\infty, +\infty]$ be locally integrable on Ω.

The distributional Laplacian of u is the linear functional $L_u : C_c^\infty(\Omega) \to \mathbb{R}$ defined by

$$L_u(f) := \int_\Omega u(x)\, \Delta f(x)\, d\lambda^N(x) \quad \forall f \in C_c^\infty(\Omega).$$

6.2 Theorem:

(i) If $u \in C^2(\Omega)$, then

$$L_u(f) = \int_\Omega \Delta u(x)\, f(x)\, d\lambda^N(x) \quad \forall f \in C_c^\infty(\Omega)$$

(ii) If $h \in H(\Omega)$, then $L_h \equiv 0$ on $C_c^\infty(\Omega)$.

(iii) If $s \in S(\Omega)$, then L_s is a positive linear functional on $C_c^\infty(\Omega)$.

<u>Proof</u>:

(i) This follows from Green's identity (Exercise 4B-1).

(ii) This is an immediate consequence of (i).

(iii) Take any $f \in C_c^\infty(\Omega)$ satisfying $f \geq 0$. Let $\emptyset \neq U \subseteq \mathbb{R}^N$ be open such that $\operatorname{supp} f \subset U$ and $\overline{U} \subset \Omega$. By Theorem 5.12, there exists a sequence

$(s_n)_{n=1}^{\infty}$ in $\mathcal{S}(U) \cap C^{\infty}(U)$ which is pointwise decreasing on U and convergent to s.

By Corollary 5.11, $\Delta s_n \geq 0$ on U, and thus, by (i),

$$\int_U s_n(x)\, \Delta f(x)\, d\lambda^N(x) = \int_U f(x)\, \Delta s_n(x)\, d\lambda^N(x) \geq 0.$$

By monotone convergence of $(s_n (\Delta f)^{\pm})_{n=1}^{\infty}$ on U, it follows that

$$L_s(f) = \int_U s(x)\,\Delta f(x)\,d\lambda^N(x)$$

$$= \lim_{n\to\infty} \int_U s_n(x)\,\Delta f(x)\,d\lambda^N(x) \geq 0,$$

as desired.

$\square$

A measure μ on $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ is a **Radon measure** if $\mu(K) < \infty$ for all compact subsets $K \subset \Omega$.

6.3 Theorem:

Let $s \in S(\Omega)$, then there exists a unique Radon measure μ_s on Ω, called the Riesz measure associated with s, such that

$$L_s(f) = \int_\Omega f(x)\, d\mu_s(x) \qquad \forall f \in C_c^\infty(\Omega).$$

Proof:

1. Claim: L_s extends to a positive linear functional $\hat{L}_s : C_c(\Omega) \to \mathbb{R}$.

By convolution with smooth functions, one can show that

$$\forall f \in C_c(\Omega) \ \exists (f_n)_{n=1}^{\infty} \text{ in } C_c^{\infty}(\Omega): \quad \sup_{x \in \Omega} |f(x) - f_n(x)| \xrightarrow{n \to \infty} 0. \tag{6.1}$$

Therefore, $(L_s(f_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$; we put

$$\hat{L}_s(f) := \lim_{n \to \infty} L_s(f_n).$$

This value is independent of the particular choice of $(f_n)_{n=1}^{\infty}$; thus, we obtain the desired extension $\hat{L}_S : C_c(\Omega) \to \mathbb{R}$ in this way.

② By the Riesz representation theorem, there exists a (unique) Radon measure μ_S on Ω such that

$$\hat{L}_S(f) = \int_{\Omega} f(x)\, d\mu_S(x) \qquad \forall f \in C_c(\Omega).$$

③ <u>Uniqueness</u>: Suppose that μ_1, μ_2 are Radon measures on Ω such that for all $f \in C_c^\infty(\Omega)$

$$\int_\Omega f(x)\, d\mu_1(x) = \int_\Omega f(x)\, d\mu_2(x), \tag{6.2}$$

then $\mu_1 = \mu_2$

By (6.1), we find that (6.2) extends to hold for all $f \in C_c(\Omega)$. The uniqueness part of the Riesz representation theorem yields then $\mu_1 = \mu_2$.

$\square$

For $u \in U(\Omega)$, we define the Riesz measure μ_u

associated with u as the Riesz measure associated with $-u \in S(\Omega)$.

Now, we can deliver on our promise which we have made at the end of Chapter 1.

6.4 Theorem:

(i) The fundamental harmonic function U_y for $\mathbb{R}^N$ with pole at $y \in \mathbb{R}^N$ (see Definition 2.4) extends by $U_y(y) := \infty$ to a superharmonic function on $\mathbb{R}^N$. Its Riesz measure is given by

$$\mu_{U_y} = a_N \delta_y, \qquad a_N := \max\{1, N-2\} N \omega_N,$$

where δ_y is the Dirac measure with atom at y.

(ii) Let μ be a finite measure on $\mathbb{R}^N$ whose support

$$\operatorname{supp} \mu := \{ x \in \mathbb{R}^N \mid \forall \varepsilon > 0 : \mu(B(x, \varepsilon)) > 0 \}$$

is compact. Then the potential Φ_μ associated with μ, which is defined by

$$\Phi_\mu(x) := \int_{\mathbb{R}^N} U_y(x) \, d\mu(y), \qquad x \in \mathbb{R}^N,$$

is superharmonic on $\mathbb{R}^N$ and harmonic on

$\mathbb{R}^N \setminus \operatorname{supp}\mu$. The Riesz measure of Φ_μ is given by

$$\mu_{\Phi_\mu} = a_N \mu .$$

<u>Proof</u>

(i) By Theorem 2.3, U_y is harmonic on $\mathbb{R}^N \setminus \{y\}$.

Using (vi) $\Rightarrow$ (i) in Theorem 5.8, it follows that $U_y \in \mathcal{U}(\mathbb{R}^N)$.

For $\mu_{U_y} = a_N \delta_y$, it suffices to show that

$$L_{-u_y}(f) = a_n f(y) \qquad \forall f \in C_c^\infty(\mathbb{R}^n)$$

Take $f \in C_c^\infty(\mathbb{R}^n)$ and choose $r > 0$ such that

$$\operatorname{supp} f \subset B(y, r)$$

By Green's identity (Ex. 4B-1), we get for $\varepsilon > 0$ that

$$\int_{A(y;\varepsilon,r)} u_y(x) \Delta f(x) \, d\lambda^n(x) = -(I_\varepsilon^1 - I_\varepsilon^2),$$

when $\bullet$ $I_\varepsilon^1 := \int_{\partial B(y,\varepsilon)} u_y(x) \langle \nabla f(x), n(x) \rangle \, d\sigma(x)$

- $$I_\varepsilon^2 := \int_{\partial B(y,\varepsilon)} f(x) \langle \nabla U_y(x), n(x) \rangle \, d\sigma(x)$$

with $\sigma := \sigma_{\partial B(y,\varepsilon)}$ and $n(x) = \dfrac{x-y}{\|x-y\|}$.

① Since for all $x \in \mathbb{R}^N \setminus \{y\}$

$$\nabla U_y(x) = -\max\{1, N-2\} \frac{n(x)}{\|x-y\|^{N-1}},$$

we infer that, as $\varepsilon \downarrow 0$,

$$I_\varepsilon^2 = -a_N \underbrace{\frac{1}{N\omega_N \varepsilon^{N-1}} \int_{\partial B(y,\varepsilon)} f(x) \, d\sigma(x)}_{\to f(y)} \longrightarrow -a_N f(y).$$

② Since for all $x \in \partial B(y, \varepsilon)$

$$U_y(x) = \begin{cases} -\log(\varepsilon) & \text{if } N = 2 \\ \varepsilon^{2-N} & \text{if } N \geq 3 \end{cases},$$

we infer (with the help of Cauchy – Schwarz)

$$|I_\varepsilon^1| \leq \max_{x \in \partial B(y,\varepsilon)} \|\nabla f(x)\| \underbrace{\int_{\partial B(y,\varepsilon)} U_y(x)\, d\sigma(x)}_{\to 0 \text{ as } \varepsilon \downarrow 0}$$

and hence $I_\varepsilon^1 \to 0$ as $\varepsilon \downarrow 0$.

In summary, we get that

$$L_{-u_y}(f) = -\lim_{\varepsilon \downarrow 0} \int_{A(y;\varepsilon,r)} u_y(x)\,\Delta f(x)\,d\lambda^N(x)$$

$$= a_N\, f(y)$$

(ii) Due to Theorem 5.8, it suffices to check that

$\Phi_\mu|_{B(x_0,r)} \in \mathcal{U}(B(x_0,r))$ for all $x_0 \in \mathbb{R}^N$ and $r > 0$

such that $B(x_0,r) \cap (\mathbb{R}^N \setminus \operatorname{supp} \mu) \neq \emptyset$.

For every such ball $B(x_0,r)$, we apply Theorem 5.14

to

$$f : B(x_0, r) \times \Omega \to (-\infty, +\infty], \quad f(x,y) := u_y(x);$$

since $\Phi_\mu(x) < \infty$ for each $x \in \mathbb{R}^N \setminus \operatorname{supp} \mu$, this gives $\Phi_\mu|_{B(x_0,r)} \in \mathcal{U}(B(x_0,r))$.

If $B(x_0, r) \subseteq \mathbb{R}^N \setminus \operatorname{supp} \mu$, then we can apply this argument to $-u_y$, which gives $\Phi_\mu|_{B(x_0,r)} \in \mathcal{H}(B(x_0,r))$.

In order to prove $\mu_{\Phi_\mu} = a_N \mu$, we must show that

$$L_{-\Phi_\mu}(f) = a_N \int_{\mathbb{R}^N} f(y)\, d\mu(y) \quad \forall f \in C_c^\infty(\mathbb{R}^N)$$

This follows with the help of Fubini's theorem from (i):

$$L_{-\Phi_\mu}(f) = -\int_{\mathbb{R}^N} \Phi_\mu(x)\, \Delta f(x)\, d\lambda^N(x)$$

$$= -\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u_y(x)\, d\mu(y) \right) \Delta f(x)\, d\lambda^N(x)$$

$$= \int_{\mathbb{R}^N} \underbrace{\left(-\int_{\mathbb{R}^N} u_y(x)\, \Delta f(x)\, d\lambda^N(x) \right)}_{= L_{-u_y}(f) = a_N f(y)} d\mu(y)$$

$$= a_N \int_{\mathbb{R}^N} f(y)\, d\mu(y).$$

$\square$

7. Logarithmic potentials

From now on, we draw our attention to the case $N=2$; we identify $\mathbb{R}^2 = \mathbb{C}$.

If μ is a finite (Borel) measure on $\mathbb{C}$ with compact support $K := \operatorname{supp}\mu$, then we refer to the potential

$$\Phi_\mu : \mathbb{C} \to (-\infty, +\infty], \quad \Phi_\mu(z) := -\int_K \log|z-w|\, d\mu(w),$$

which was defined in Theorem 6.4 (ii), as the logarithmic potential associated with μ; Φ_μ is superharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus K$.

7.1 Theorem:

Let μ be a finite measure on $\mathbb{C}$ with compact support $K = \operatorname{supp} \mu$.

(i) $\Phi_\mu(z) = -\mu(\mathbb{C}) \log |z| + O\left(\frac{1}{|z|}\right)$ as $z \to \infty$

(ii) Let $w_0 \in K$. Then

$$\limsup_{z \to w_0} \Phi_\mu(z) = \limsup_{K \ni w \to w_0} \Phi_\mu(w).$$

Further, if

$$\lim_{K \ni w \to w_0} \Phi_\mu(w) = \Phi_\mu(w_0),$$

then $\displaystyle\lim_{z \to w_0} \Phi_\mu(z) = \Phi_\mu(w_0)$.

(iii) If $\mu \in \mathbb{R}$ is such that

$$\Phi_\mu(w) \leq M \qquad \forall w \in K$$

then $$\Phi_\mu(z) \leq M \qquad \forall z \in \mathbb{C}.$$

7.2. Definition:

Let μ be a finite measure on $\mathbb{C}$ with compact support K. We call

$$I(\mu) := \int_K \Phi_\mu(z)\, d\mu(z) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w)$$

the energy of μ.

7.3 Definition:

(i) A subset $E \subseteq \mathbb{C}$ is called **polar** if $I(\mu) = \infty$ for every finite measure $\mu \neq 0$ with compact support $\operatorname{supp} \mu \subseteq E$.

(ii) A property is said to hold **nearly everywhere (n.e.)** on $S \subseteq \mathbb{C}$ if it holds on $S \setminus E$ for some Borel polar set E.

7.4. Theorem:

(i) If μ is a finite measure on $\mathbb{C}$ with compact support satisfying $I(\mu) < \infty$, then $\mu(E) = 0$ for every Borel polar set $E \subseteq \mathbb{C}$

(ii) Every Borel polar set $E \subseteq \mathbb{C}$ satisfies $\lambda^2(E) = 0$; in particular

"nearly everywhere" $\Longrightarrow$ "almost everywhere".

(iii) A countable union of Borel polar sets is polar.

7.5 Definition:

Let $K \subset \mathbb{C}$ be compact. We denote by $\mathcal{P}(K)$ the set of all probability measures on K (i.e., measures $\mu : \mathcal{B}(K) \to [0,1]$ with $\mu(K) = 1$).

If there exists $\nu \in \mathcal{P}(K)$ such that

$$I(\nu) = \inf_{\mu \in \mathcal{P}(K)} I(\mu),$$

then ν is called an equilibrium measure for K.

7.6 Theorem:

Every compact set $K \subset \mathbb{C}$ has an equilibrium measure.

<u>Proof</u> (sketch):

① If $(\mu_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{P}(K)$ which is weak*-convergent to some $\mu \in \mathcal{P}(K)$, i.e.,

$$\int_K f(z)\, d\mu_n(z) \xrightarrow{n\to\infty} \int_K f(z)\, d\mu(z) \quad \forall f \in C(K),$$

then $\liminf_{n\to\infty} I(\mu_n) \geq I(\mu)$.

② Every sequence in $\mathcal{P}(K)$ has a weak*-convergent subsequence.

Having this, we choose a sequence $(\mu_n)_{n=1}^{\infty}$ in $\mathcal{P}(K)$ such that

$$I(\mu_n) \xrightarrow{n\to\infty} \inf_{\mu\in\mathcal{P}(K)} I(\mu) \qquad (7.1)$$

By ②, $(\mu_n)_{n=1}^{\infty}$ has a subsequence, say $(\mu_{n_k})_{k=1}^{\infty}$, which is weak*-convergent to some $\nu \in \mathcal{P}(K)$. Then

$$\inf_{\mu \in \mathcal{P}(K)} I(\mu) \le I(\nu) \overset{①}{\le} \liminf_{n\to\infty} I(\mu_{n_k}) \overset{(7.1)}{=} \inf_{\mu \in \mathcal{P}(K)} I(\mu),$$

i.e., ν is an equilibrium measure for K.

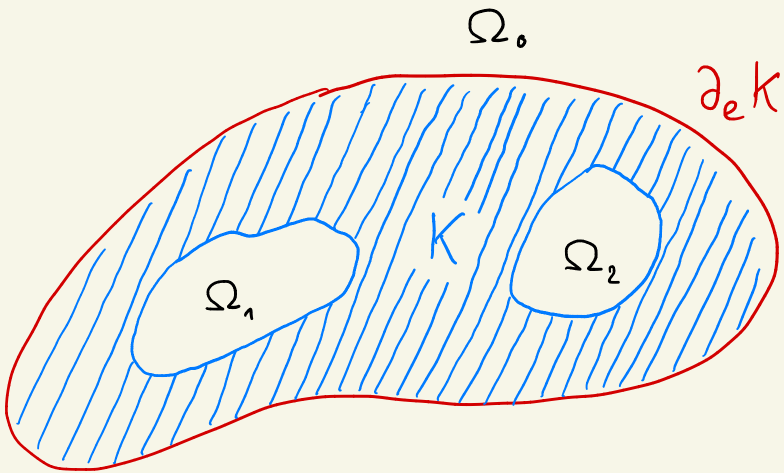
$\square$

7.7 Remark:

If $K \subset \mathbb{C}$ is compact and not polar, then there is a unique equilibrium measure ν_K for K.

Moreover, we have that $\operatorname{supp} \nu_K \subseteq \partial_e K$, where $\partial_e K$

denotes the exterior boundary of K, i.e., the boundary of the unbounded connected component of $\mathbb{C} \setminus K$.



$\Omega_0 \cup \Omega_1 \cup \Omega_2 = \mathbb{C} \setminus K$

Ω_0 unbounded

7.8 Theorem (Frostman)

Let $K \subset \mathbb{C}$ be compact and let ν be an equilibrium measure for K. Then:

(i) $\forall z \in \mathbb{C}: \quad \Phi_\nu(z) \leq I(\nu)$

(ii) $\exists E \subseteq \partial K$ polar $\forall z \in K \setminus E: \quad \Phi_\nu(z) = I(\nu)$.

7.9 Definition:

Let $E \subseteq \mathbb{C}$ be any subset. We call

$$\operatorname{cap}(E) := \sup_{\mu \in \mathcal{P}(E)} e^{-I(\mu)}$$

the logarithmic capacity of E. Here $\mathcal{P}(E)$ denotes the set of all Borel probability measures μ on $\mathbb{C}$ with compact support satisfying $\operatorname{supp} \mu \subseteq E$.

<u>Note:</u> If $K \subset \mathbb{C}$ is compact and ν is an equilibrium measure for K, then

$$\operatorname{cap}(K) = e^{-I(\nu)}.$$

8. Uniform approximation

Suppose that $K \subset \mathbb{C}$ is a compact set for which $\mathbb{C} \setminus K$ is connected. In this situation, Runge's theorem says that every $f \in \mathcal{O}(\Omega)$ on some open set $\Omega \subseteq \mathbb{C}$ satisfying $K \subset \Omega$ can be approximated uniformly on K by (holomorphic) polynomials. We prove here a quantitative version of this result.

8.1 Theorem (Bernstein - Walsh):

Let $K \subset \mathbb{C}$ be compact and suppose that $\mathbb{C} \setminus K$ is connected; let $\nu \in \mathcal{P}(K)$ be an equilibrium measure for K. Suppose that $f \in \mathcal{O}(\Omega)$, where $K \subset \Omega \subseteq \mathbb{C}$ is open. Put

$$\Theta := \begin{cases} \sup\limits_{z \in (\mathbb{C} \cup \{\infty\}) \setminus \Omega} e^{\Phi_\nu(z) - I(\nu)} & , \text{ if } \operatorname{cap}(K) > 0 \\ 0 & , \text{ if } \operatorname{cap}(K) = 0 \end{cases}$$

Then $\Theta < 1$ and

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} \leq \Theta ,$$

where

$$d_n(f,K) := \inf \{ \|f-p\|_K \mid p \text{ hol. poly.}, \deg p \leq n \}$$

The proof relies on the following result.

8.2 Theorem (Bernstein's lemma)

In the situation of Theorem 8.1, let $\operatorname{cap}(K) > 0$.

Then:

(i) If q is a polynomial with $n := \deg q \geq 1$, then

$$\left(\frac{|q(z)|}{\|q\|_K} \right)^{1/n} \leq e^{-\Phi_\nu(z) + I(\nu)} \qquad \forall z \in \mathbb{C} \setminus K.$$

(ii) If q is a Fekete polynomial for K with

$n := \deg q \geq 2$, then

$$\left(\frac{|q(z)|}{\|q\|_K}\right)^{1/n} \geq e^{-\Phi_\nu(z) + I(\nu)} \left(\frac{\operatorname{cap}(K)}{\delta_n(K)}\right)^{\tau(z,\infty)} \quad \forall z \in \mathbb{C}\setminus K,$$

where $\tau := \tau_{\Omega(K)}$ is the Harnack distance on $\Omega(K) = (\mathbb{C} \cup \{\infty\}) \setminus K$.

8.3 Remark:

(i) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and connected. Building on Ex-3A-1, we define for $x, y \in \Omega$

$$\tau_\Omega(x,y) := \inf\{\tau > 0 \mid \forall u \in H_+(\Omega) : \\ \tau^{-1}u(x) \le u(y) \le \tau u(x)\}$$

We call $\tau_\Omega : \Omega \times \Omega \to [1,\infty)$ the Harnack distance on Ω; one can show that

$$\log \tau_\Omega : \Omega \times \Omega \to [0,\infty)$$

is a continuous semimetric on Ω.

(ii) Due to Theorem 5.13, one can extend the notions of sub-, super-, and harmonic functions to Riemann surfaces and in particular to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For instance, the Harnack distance and the maximum principle (Theorems 3.8 and 5.7) remain true.

Proof of Theorem 8.2:

(i) With no loss of generality, we may suppose that q is monic (i.e., $q(z) = z^n + a_{n-1} z^{n-1} + \dots$). By

$$u(z) := \frac{1}{n} \log |q(z)| - \frac{1}{n} \log \|q\|_K + \Phi_\nu(z) - I(\nu)$$

for all $z \in \mathbb{C} \setminus K$, we define a function $u \in S(\mathbb{C} \setminus K)$. (Note that $z \mapsto \frac{1}{n} \log |q(z)|$ is subharmonic on $\mathbb{C} \setminus K$; see Ex 1B-1 and Thm 5.8)

Since $\Phi_\nu(z) = -\log|z| + O\left(\frac{1}{|z|}\right)$ (see Thm 7.1) and $\frac{1}{n} \log |q(z)| = \log |z| + o(1)$ as $z \to \infty$, we see that u extends by $u(\infty) := -I(\nu) - \frac{1}{n} \log \|q\|_K$

to a function $u \in S(\Omega(K))$.

Now, for every $w \in \partial K$, we get by Thm 7.8 (i)

$$\limsup_{z \to w} u(z) \leq \frac{1}{n} \log |q(w)| - \frac{1}{n} \log \|q\|_K \leq 0$$

By Thm 5.7 (iii) (see Rem 8.3 (ii)), it follows that $u \leq 0$ on $\Omega(K)$, which implies the assertion.

(ii) Note that all zeros of q lie in K; thus, u

is harmonic on $\Omega(K)$ and $u \leq 0$. Hence

$-u \in H_+(\Omega(K))$ and Rem 8.3 (i) gives

$$u(z) \geq \tau_{\Omega(K)}(z, \infty)\, u(\infty) \qquad \forall z \in \Omega(K).$$

By Ex 4B-2(ii), we have

$$u(\infty) = -I(\nu) - \frac{1}{n} \log \|q\|_K$$

$$\geq -I(\nu) - \log \delta_n(K) = \log\left(\frac{\operatorname{cap}(K)}{\delta_n(K)}\right)$$

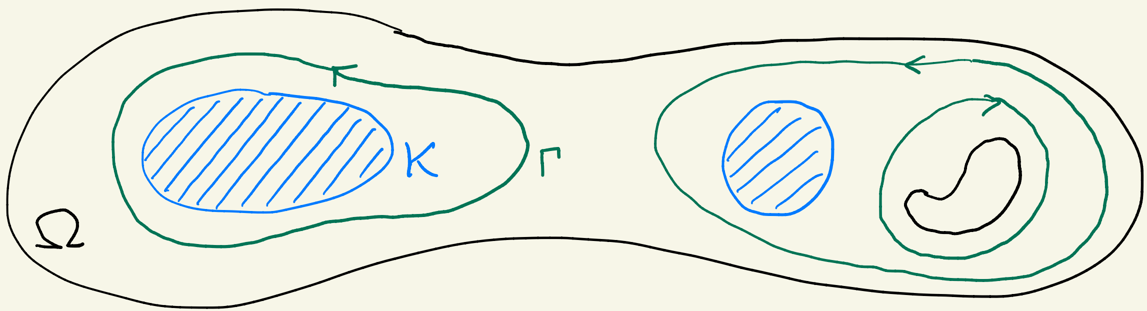
Putting this together, we obtain the result. $\square$

Proof of Theorem 8.1:

Suppose that $\operatorname{cap}(K) > 0$. Let Γ be a closed contour in $\Omega \setminus K$ such that

$$\operatorname{Ind}_\Gamma(w) = 1 \quad \forall w \in K \qquad \text{and}$$

$$\operatorname{Ind}_\Gamma(z) = 0 \quad \forall z \in \mathbb{C} \setminus \Omega.$$



By the global version of Cauchy's integral formula (Satz 7.12, FTI), we have

$$(8.1) \qquad f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w}\, dz \qquad \forall w \in K$$

For $n \geq 2$, let q_n be a Fekete polynomial of degree n for K and put

$$(8.2) \qquad p_n(w) := \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{q_n(z)} \, \frac{q_n(w) - q_n(z)}{w - z}\, dz .$$

Then p_n is a polynomial with $\deg p_n \leq n-1$.

From (8.1) and (8.2), we deduce that

$$f(w) - p_n(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w} \frac{q_n(w)}{q_n(z)} \, dz \quad \forall w \in K$$

Hence,

$$d_n(f, K) \leq \| f - p_n \|_K \leq C \frac{\| q_n \|_K}{\inf\limits_{z \in \Gamma} |q_n(z)|},$$

where $C := \frac{L(\Gamma)}{2\pi} \cdot \| f \|_\Gamma \operatorname{dist}(\Gamma, K)^{-1}$. By Thm 8.2(iii)

$$\left(\frac{\|q_n\|_K}{|q_n(z)|}\right)^{1/n} \leq e^{\Phi_\nu(z) - I(\nu)} \Bigg(\underbrace{\frac{\delta_n(K)}{\mathrm{cap}(K)}}_{\geq 1 \text{ by Theorem 7.11}}\Bigg)^{\tau_{\Omega(K)}(z,\infty)}$$

$$\leq \Theta_\Gamma \cdot \left(\frac{\delta_n(K)}{\mathrm{cap}(K)}\right)^{\alpha} \qquad \forall z \in \Gamma,$$

where

$$\Theta_\Gamma := \sup_{z \in \Gamma} e^{\Phi_\nu(z) - I(\nu)} \qquad \text{and}$$

$$\alpha := \sup_{z \in \Gamma} \tau_{\Omega(K)}(z,\infty).$$

Hence, due to Theorem 7.11,

$$\limsup_{n\to\infty} d_n(f,K)^{1/n} \le \limsup_{n\to\infty} c^{1/n}\, \Theta_\Gamma \left(\frac{\delta_n(K)}{\operatorname{cap}(K)}\right)^\alpha$$

$$= \Theta_\Gamma .$$

Finally, we note that

$$\forall \varepsilon > 0 \quad \exists \Gamma \text{ as above} : \quad 0 \le \Theta_\Gamma - \Theta < \varepsilon$$

This proves $\limsup_{n\to\infty} d_n(f,K)^{1/n} \le \Theta$ if $\operatorname{cap}(K) > 0$.

Note that $\Theta < 1$, because otherwise, there was a $z_0 \in \Gamma \subset \mathbb{C} \setminus K$ such that $\Phi_\nu(z_0) = I(\nu)$; thus, by Theorem 7.8 (i), z_0 was a local maximum, so that $\Phi_\nu \equiv I(\nu)$ on $\mathbb{C} \setminus K$ by Theorem 5.7 (ii), in contradiction to Theorem 7.1 (i) as $I(\nu) < \infty$.

The assertion in the case $\operatorname{cap}(K) = 0$ follows by approximation of K with a decreasing sequence

$(K_k)_{k=1}^{\infty}$ of non-polar compact subsets of Ω

satisfying $K = \bigcap_{k=1}^{\infty} K_k$ from the already

proved case. $\square$