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FACHRICHTUNG MATHEMATIK
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Additional material for the lecture
Potential Theory in the Complex Plane
Summer term 2020

Remark 5.15. Subharmonic functions have some remarkable applications in functional analysis.
Let $(A, \|\cdot\|)$ be a unital complex Banach algebra. For $a \in A$, we define the *spectral radius* $r(a)$ of a by

$$r(a) := \max\{|z| \mid z \in \sigma(a)\},$$

where

$$\sigma(a) := \{z \in \mathbb{C} \mid a - z\mathbf{1}_A \text{ not invertible in } A\}$$

is the *spectrum* of A; recall that we have the formula

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$

Vesentini's theorem says that, for every holomorphic function $f : \Omega \to A$ on an open set $\emptyset \neq \Omega \subseteq \mathbb{C}$,

$$s:\ \Omega \longrightarrow [-\infty, +\infty), \quad z \longmapsto \log r(f(z))$$

is subharmonic, provided that $s \not\equiv -\infty$ on each connected component of Ω.
Using this remarkable fact, one can prove Johnson's theorem which says that a surjective homomorphism $\theta : A_1 \to A_2$ between complex Banach algebras A_1 and A_2 is automatically continuous if A_2 is semisimple (i.e., the intersection of all maximal right ideals in A_2 is $\{0\}$).

Remark 6.5. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be in $C_c^k(\mathbb{R}^N) = C_c(\mathbb{R}^N) \cap C^k(\mathbb{R}^N)$ for some $k \geq 1$. It can be shown that

$$\Phi(x) := \int_{\mathbb{R}^N} U_y(x)\rho(y)\, \mathrm{d}\lambda^N(y) \qquad \text{for } x \in \mathbb{R}^N$$

defines a function $\Phi \in C^{k+1}(\mathbb{R}^N)$.
Further, Φ solves *Poisson's equation* $\Delta\Phi = -a_N\rho$ on $\mathbb{R}^N$. Indeed, by Theorem 6.2 (i), we have for all $f \in C_c^\infty(\mathbb{R}^N)$,

$$L_\Phi(f) = \int_{\mathbb{R}^N} f(x)\Delta\Phi(x)\, \mathrm{d}\lambda^N(x),$$

whereas, for the measures $\mu^\pm$ defined by $\mathrm{d}\mu^\pm(y) := \rho^\pm(y)\,\mathrm{d}\lambda^N(y)$, Theorem 6.4 (ii) yields that

$$\begin{aligned} L_\Phi(f) &= L_{\Phi_{\mu^+}}(f) - L_{\Phi_{\mu^-}}(f) \\ &= -a_N \int_{\mathbb{R}^N} f(x)\,\mathrm{d}\mu^+(x) + a_N \int_{\mathbb{R}^N} f(x)\,\mathrm{d}\mu^-(x) \\ &= -a_N \int_{\mathbb{R}^N} f(x)\rho(x)\,\mathrm{d}\lambda^N(x), \end{aligned}$$

where the first step relies on the decomposition $\Phi = \Phi_{\mu^+} - \Phi_{\mu^-}$ which is valid thanks to $\rho = \rho^+ - \rho^-$. Hence, in summary, for all $f \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(x)\big(\Delta\Phi(x) + a_N\rho(x)\big)\,\mathrm{d}\lambda^N(x) = 0,$$

which implies $\Delta\Phi(x) = -a_N\rho(x)$ for all $x \in \mathbb{R}^N$.
In particular, for $N = 3$, since $a_3 = 4\pi$, we get $\Delta\Phi = -4\pi\rho$. In the notation of Chapter 1, we have $\operatorname{grad}\Phi = -4\pi\varepsilon_0\vec{E}$, so that the latter identity yields $\operatorname{div}\vec{E} = \frac{1}{\varepsilon_0}\rho$, which is the differential form of Gauss' law.

Theorem 7.10.

(i) *For $E \subseteq \mathbb{C}$, we have $\operatorname{cap}(E) = 0$ if and only if E is polar.*

(ii) *If $E_1 \subseteq E_2 \subseteq \mathbb{C}$, then $\operatorname{cap}(E_1) \leq \operatorname{cap}(E_2)$.*

(iii) *If $E \subseteq \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$, then $\operatorname{cap}(\alpha E + \beta) = |\alpha| \operatorname{cap}(E)$.*

(iv) *If $K \subset \mathbb{C}$ is compact, then $\operatorname{cap}(K) = \operatorname{cap}(\partial_e K)$.*

(v) *For a compact set $K \subset \mathbb{C}$, we denote by $\Omega(K)$ the connected component of $(\mathbb{C} \cup \{\infty\}) \setminus K$ which contains ∞.*

If $K_1, K_2 \subset \mathbb{C}$ are compact and $f : \Omega(K_1) \to \Omega(K_2)$ is a meromorphic function satisfying $f(z) = z + O(1)$ as $z \to \infty$, then $\operatorname{cap}(K_2) \leq \operatorname{cap}(K_1)$; if f is biholomorphic, then $\operatorname{cap}(K_2) = \operatorname{cap}(K_1)$.

(vi) *If $K \subset \mathbb{C}$ is compact, then*

$$\operatorname{cap}(K) \leq \frac{1}{2}\operatorname{diam}(K) \qquad \textit{and} \qquad \operatorname{cap}(K) \geq \sqrt{\frac{1}{\pi}\lambda^2(K)},$$

where $\operatorname{diam}(K) := \max\{|w_1 - w_2| \mid w_1, w_2 \in K\}$ and λ^2 denotes the Lebesgue measure on $\mathbb{C}$.

(vii) *If $K \subset \mathbb{C}$ is compact and $q(z) = \sum_{k=0}^d a_k z^k$ with $a_d \neq 0$ a complex polynomial, then*

$$\operatorname{cap}(q^{-1}(K)) = \Big(\frac{\operatorname{cap}(K)}{|a_d|}\Big)^{1/d}.$$

Theorem 7.11 (Fekete-Szegö)**.** *Let $K \subset \mathbb{C}$ be compact. Consider the sequence $(\delta_n(K))_{n=2}^\infty$ of diameters of K, which was defined in Exercise 4B-2. Then $(\delta_n(K))_{n=2}^\infty$ is convergent and the limit $\delta(K) := \lim_{n\to\infty}\delta_n(K)$ is given by $\delta(K) = \operatorname{cap}(K)$.*

Proof. From Exercise 4B-2 (i), we know that $(\delta_n(K))_{n=2}^{\infty}$ is decreasing; since $\delta_n(K) \geq 0$ for all $n \geq 2$, it follows that $(\delta_n(K))_{n=2}^{\infty}$ is convergent.

① **Claim:** For all $n \geq 2$, it holds that $\delta_n(K) \geq \operatorname{cap}(K)$.

For $w_1, \dots, w_n \in K$, we have by definition of $\delta_n(K)$ that

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \log |w_i - w_j| \leq \log \delta_n(K).$$

Hence, for every $\mu \in \mathcal{P}(K)$, we get by integration of the latter inequality with respect to the product measure μ^n over K^n

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \int_K \cdots \int_K \log |w_i - w_j| \,\mathrm{d}\mu(w_1) \cdots \mathrm{d}\mu(w_n) \leq \log \delta_n(K).$$

Since for each of the $\frac{n(n-1)}{2}$ possible choices of indices $1 \leq i < j \leq n$

$$\int_K \cdots \int_K \log |w_i - w_j| \,\mathrm{d}\mu(w_1) \cdots \mathrm{d}\mu(w_n) = \int_K \int_K \log |w_i - w_j| \,\mathrm{d}\mu(w_i) \,\mathrm{d}\mu(w_j) = -I(\mu),$$

we infer from the latter that $e^{-I(\mu)} \leq \delta_n(K)$. Thus, if follows that

$$\operatorname{cap}(K) = \sup_{\mu \in \mathcal{P}(K)} e^{-I(\mu)} \leq \delta_n(K),$$

as desired.

② **Claim:** For each $n \geq 2$, let $w^{(n)} = (w_1^{(n)}, \dots, w_n^{(n)})$ be a Fekete n-tuple for K and define $\mu_n \in \mathcal{P}(K)$ by

$$\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{w_i^{(n)}}.$$

Let $(\mu_{n_k})_{k=1}^{\infty}$ be a subsequence of $(\mu_n)_{n=1}^{\infty}$ which is weak*-convergent to some $\nu \in \mathcal{P}(K)$. Then $I(\nu) \leq -\log \delta(K)$.

For $R > 0$, we set $\log_R(x) := \min\{\log(x), R\}$. Then, by monotone convergence,

$$I(\nu) = \lim_{R \to \infty} \int_K \int_K \log_R \frac{1}{|z-w|} \,\mathrm{d}\nu(z) \,\mathrm{d}\nu(w)$$

and thus, since $(\mu_{n_k})_{k=1}^{\infty}$ is weak*-convergent to ν,

$$I(\nu) = \lim_{R \to \infty} \lim_{k \to \infty} \int_K \int_K \log_R \frac{1}{|z-w|} \,\mathrm{d}\mu_{n_k}(z) \,\mathrm{d}\mu_{n_k}(w).$$

Next, we observe that

$$\begin{aligned}
\int_K \int_K \log_R \frac{1}{|z-w|} \,\mathrm{d}\mu_{n_k}(z) \,\mathrm{d}\mu_{n_k}(w) &= \frac{1}{n_k^2} \sum_{i,j=1}^{n} \log_R \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|} \\
&= \frac{2}{n_k^2} \sum_{1 \leq i < j \leq n}^{n} \log_R \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|} + \frac{1}{n_k^2} n_k R \\
&\leq \frac{2}{n_k^2} \sum_{1 \leq i < j \leq n}^{n} \log \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|} + \frac{R}{n_k} \\
&= -\frac{n_k - 1}{n_k} \log \delta_{n_k}(K) + \frac{R}{n_k}.
\end{aligned}$$

Hence, we deduce that

$$I(\nu) \leq \lim_{R\to\infty} \lim_{k\to\infty} \Big(-\frac{n_k - 1}{n_k} \log \delta_{n_k}(K) + \frac{R}{n_k} \Big) = -\log \delta(K),$$

as asserted.

③ Combining the results derived above, we obtain that

$$\operatorname{cap}(K) \overset{①}{\leq} \delta(K) \overset{②}{\leq} e^{-I(\nu)} \leq \sup_{\mu\in\mathcal{P}(K)} e^{-I(\mu)} = \operatorname{cap}(K),$$

i.e., $\delta(K) = \operatorname{cap}(K)$, which proves the theorem.
Further, we see that ν must be an equilibrium measure for K. As there is a unique equilibrium measure ν_K for K in the case $\operatorname{cap}(K) > 0$, it follows from ② that the sequence $(\mu_n)_{n=1}^{\infty}$ then has ν_K as its only limit point; therefore, $(\mu_n)_{n=1}^{\infty}$ itself must be weak*-convergent to ν_K. □

Remark 8.4. We notice that the polynomials p_n defined in (8.2) satisfy for $j = 1, \dots, n$

$$p_n(w_j) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{q_n(z)} \frac{q_n(w_j) - q_n(z)}{w_j - z} \,\mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w_j} \,\mathrm{d}z = f(w_j),$$

where, in the second step, we have used that $q_n(w_j) = 0$, and, in the last step, Cauchy's integral formula as formulated in (8.1). In other words, p_n solves the following interpolation problem:

> Find a holomorphic complex polynomial p with $\deg p \leq n-1$ such that
>
> $$p(w_j) = f(w_j) \qquad \text{for } j = 1, \dots, n.$$

Note that if $w_1, \dots, w_n$ are all distinct, then p_n is the unique solution of this interpolation problem. In this case, one can use the so-called *Lagrange polynomials* to find and explicit expression for p_n.
Among all holomorphic complex polynomials p satisfying $\deg p \leq n$, there is always at least one *best approximation* p_* to f, i.e., p_* satisfies the condition $d_n(f, K) = \|f - p_*\|_K$. In general, the polynomials p_n defined in (8.2) do not provide best approximations to f. Therefore, it seems possible that a better choice of p_n might lead to better results about the asymptotic behavior of $d_n(f, K)$ as $n \to \infty$. However, one can show that always

$$\|f - p_n\|_K \leq (n+1) d_n(f, K) \qquad \text{for all } n \geq 2.$$

Thus, we see that $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leq \theta$, namely the conclusion of Theorem 8.1, holds if and only if $\limsup_{n\to\infty} \|f - p_n\|_K^{1/n} \leq \theta$.

Example 8.5. Fix $z_0 \in \mathbb{C}$ and $r_0 > 0$ and put $K := \overline{D(z_0, r_0)}$. Then $\operatorname{cap}(K) = r_0$ and the (unique) equilibrium measure ν_K is given by $\nu_K = \frac{1}{2\pi r_0} \sigma_{\partial D(z_0, r_0)}$. One finds that the associated logarithmic potential Φ_{ν_K} is of the form

$$\Phi_{\nu_K}(z) = \begin{cases} \log \frac{1}{r_0}, & \text{if } |z - z_0| \leq r_0 \\ \log \frac{1}{|z - z_0|} & \text{if } |z - z_0| > r_0 \end{cases}.$$

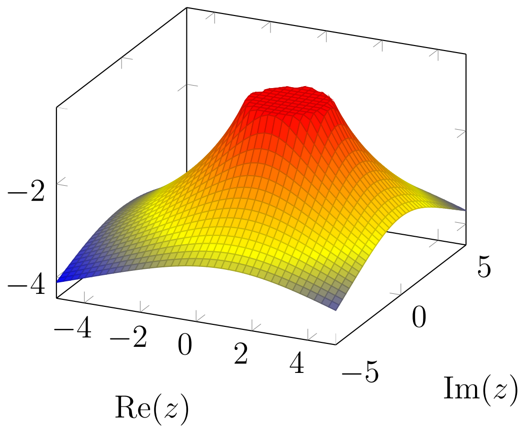

Figure 1: Graph of the potential Φ_{ν_K} for the equilibrium measure ν_K for $K = \overline{D(z_0, r_0)}$ with $z_0 = 1$ and $r_0 = 2$; see Example 8.5.

Now, for any $r > r_0 > 0$, we consider $\Omega := D(z_0, r)$. Then,

$$\theta = \sup_{z \in (\mathbb{C} \cup \{\infty\}) \setminus \Omega} e^{\Phi_{\nu_K}(z) - I(\nu)} = \frac{r_0}{r}.$$

Hence, Theorem 8.1 asserts that $\limsup_{n\to\infty} d_n(f, K)^{1/n} \leq \theta$ for every $f \in \mathcal{O}(\Omega)$. This is in accordance with the rate of approximation of f by its Taylor polynomials at the point z_0. In fact, if we put

$$T_n(w) := \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(w - z_0)^k$$

for every integer $n \geq 0$, then Cauchy's integral formula yields for every $r_0 < \rho < r$ and all $w \in K$ that

$$\begin{aligned} f(w) - T_n(w) &= \frac{1}{2\pi i} \int_{\gamma_{z_0,\rho,\circlearrowleft}} f(\zeta) \left(\frac{1}{\zeta - w} - \sum_{k=0}^{n} \frac{(w - z_0)^k}{(\zeta - z_0)^{k+1}} \right) \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_{\gamma_{z_0,\rho,\circlearrowleft}} f(\zeta) \sum_{k=n+1}^{\infty} \frac{(w - z_0)^k}{(\zeta - z_0)^{k+1}} \mathrm{d}\zeta. \end{aligned}$$

We infer from the latter that

$$\|f - T_n\|_K \leq \|f\|_{\partial D(z_0,\rho)} \frac{1}{1 - \frac{r_0}{\rho}} \left(\frac{r_0}{\rho} \right)^{n+1},$$

which yields $\limsup_{n\to\infty} \|f - T_n\|_K^{1/n} \leq \frac{r_0}{\rho}$; as $r_0 < \rho < r$ was arbitrary, we can let $\rho \nearrow r$, which gives $\limsup_{n\to\infty} \|f - T_n\|_K^{1/n} \leq \theta$.

Example 8.6. For the interval $K = [-1, 1]$, one can show that $\operatorname{cap}(K) = \frac{1}{2}$ and that the (unique) equilibrium measure is given by

$$\mathrm{d}\nu_K(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x.$$

Further, one obtains that

$$\Phi_{\nu_K}(z) = \begin{cases} \log(2) & \text{if } z \in [-1, 1] \\ \log(2) - \log\left|z + \sqrt{z^2 - 1}\right| & \text{if } z \in \mathbb{C} \setminus [-1, 1] \end{cases}.$$

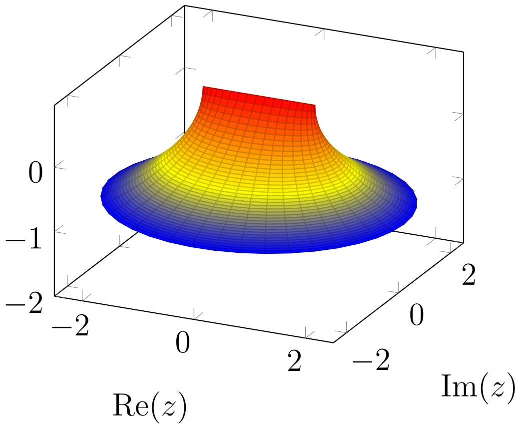

Figure 2: Graph of the potential Φ_{ν_K} for the equilibrium measure ν_K for $K = [-1, 1]$; see Example 8.6.