Potential Theory in the Complex Plane
held by Dr. Tobias Mai in Summer 2020
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General Information

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Chapter I.

A Physical Motivation of Potential Theory

Potential theory has its origins in mathematical physics of the 19th century, namely in the study of gravity and the electromagnetic force. Let us take a look at electrostatics.

Consider a (negatively) charged body \( K \subseteq \mathbb{R}^3 \). The body is surrounded by an electric field \( \vec{E} \), i.e. the force acting on a test particle with the charge \( q \) at the position \( \vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 - K \) is \( q \vec{E}(\vec{x}) \). By Coulomb’s law, a particle with the charge \( q_0 \) at the point \( \vec{x}_0 = (x^1_0, x^2_0, x^3_0) \) induces the electric field

\[
\vec{E}(\vec{x}) = \frac{q_0}{4\pi \varepsilon_0 \| \vec{x} - \vec{x}_0 \|^3} (\vec{x} - \vec{x}_0),
\]

where \( \varepsilon_0 \) is the vacuum permittivity and \( \| \vec{x} - \vec{x}_0 \| \) is the euclidean distance between \( \vec{x} \) and \( \vec{x}_0 \). At the end of the 18th century, it was observed by Lagrange that there exists a scalar-valued function \( \Phi \), called the potential of \( \vec{E} \), such that \( \vec{E} = -\nabla \Phi \), where \( \nabla \Phi(\vec{x}) = (\partial_1 \Phi(\vec{x}), \partial_2 \Phi(\vec{x}), \partial_3 \Phi(\vec{x}))^t \) is the gradient of \( \Phi \) in \( \vec{x} \) with respect to the standard basis of \( \mathbb{R}^3 \) and the euclidean inner product. Indeed,

\[
\Phi(\vec{x}) = \frac{1}{4\pi \varepsilon_0 \| \vec{x} - \vec{x}_0 \|} q_0.
\]

This law has the advantage that the work \( W \) done by the electric field \( \vec{E} \) when it moves a particle with charge \( q \) from a point \( \vec{x}_1 \) to a point \( \vec{x}_2 \) along the path \( \gamma: [t_1, t_2] \to \mathbb{R}^3 \) with \( \gamma(t_1) = \vec{x}_1 \) and \( \gamma(t_2) = \vec{x}_2 \), which by experiment is found to be

\[
W = \int_{\gamma} q \vec{E}(\vec{x}) = q \int_{t_1}^{t_2} \langle \vec{E}(\gamma(t)), \dot{\gamma}(t) \rangle \, dt,
\]

can be computed easily. Namely it holds

\[
\langle \vec{E}(\gamma(t)), \dot{\gamma}(t) \rangle = -\langle \nabla \Phi(\gamma(t)), \dot{\gamma}(t) \rangle = -(\Phi \circ \gamma)'(t)
\]
so that $W = -q(\Phi(\vec{x}_2) - \Phi(\vec{x}_1))$. Now suppose that we have charges which are “continuously” distributed over the body $K$, i.e. there is a function $\rho: K \to \mathbb{R}$, the so-called charge density, such that

$$Q_B := \int_B \rho(\vec{x}_0) \, d\vec{x}_0^1d\vec{x}_0^2d\vec{x}_0^3$$

yields the charge of the portion $B \subseteq K$. Then, the electric field surrounding the body $K$ in the point $\vec{x} \in \Omega := \mathbb{R}^3 - K$ is given by

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_K \frac{\rho(\vec{x})}{\|\vec{x} - \vec{x}_0\|^3} (\vec{x} - \vec{x}_0) \, d\vec{x}_0^1d\vec{x}_0^2d\vec{x}_0^3,$$

which has the potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_K \frac{\rho(\vec{x}_0)}{\|\vec{x} - \vec{x}_0\|} \, d\vec{x}_0^1d\vec{x}_0^2d\vec{x}_0^3.$$

One can check that $\Phi: \Omega \to \mathbb{R}$ satisfies $\Delta \Phi \equiv 0$, where $\Delta := \sum_{i=1}^3 \partial_i^2$ is the Laplace operator, i.e. $\Phi$ is a harmonic function. Potential theory explains, in particular, how $\rho$ can be recovered from (a suitable extension) of $\Phi$; this leads to the differential form of Gauss’s law.
Chapter II.

Harmonic functions

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be an open subset. Note that we suppose that $\mathbb{R}^N$ is endowed with the euclidean norm, i.e. for $x = (x^1, \ldots, x^N)$ and $y = (y^1, \ldots, y^N) \in \mathbb{R}^N$ we have

$$
\|x\| := \left( \sum_{i=1}^{N} |x_i|^2 \right)^{1/2}, \quad \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i.
$$

We denote

$$
C(\Omega) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is continuous} \},
$$

$$
C^k(\Omega) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is } k \text{ times continuously differentiable} \}
$$

and by $C^\infty(\Omega) := \bigcap_{k \geq 0} C^k(\Omega)$ we denote the space of smooth, i.e. arbitrarily often continuous differentiable functions.

**Definition II.1:** Let $f : \Omega \to \mathbb{R}$ be a function. If $f$ belongs to $C^2(\Omega)$ and solves Laplace’s equation $\partial_i^2 f + \cdots + \partial_N^2 f \equiv 0$ (i.e. $\Delta f \equiv 0$, where again $\Delta := \sum_{i=1}^{N} \partial_i^2$ : $C^2(\Omega) \to C(\Omega)$ denotes the Laplace operator), $f$ is called harmonic. We denote by $H(\Omega)$ the set of all harmonic functions on $\Omega$.

**Remark II.2:** (i) Since the Laplace operator $\Delta : C^2(\Omega) \to C(\Omega)$ is linear and $H = \ker(\Delta : C^2(\Omega) \to C(\Omega))$, it follows that $H(\Omega)$ is a vector space. It contains all (locally) affine linear functions (i.e. functions $f$ for which there are $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that for all $x \in \Omega$ it holds $f(x) = \langle x, a \rangle + b$). For $N = 1$, these are clearly all harmonic functions, thus we only consider $N \geq 2$ in the following.

(ii) If $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry (i.e. there are $Q \in O(N)$ and $a \in \mathbb{R}^N$ such that for all $x \in \Omega$ it holds $\phi(x) = a + Qx$) or an dilation (i.e. there is some positive real number $a$ such that for all $x \in \Omega$ we have $\phi(x) = ax$), then for all $f \in H(\Omega)$, the composition $f \circ \phi$ belongs to $H(\phi(\Omega))$. 

(iii) If $\Omega' \subseteq \Omega$ is an open, non-empty subset, then clearly for any $f \in H(\Omega)$, the restriction $f|_{\Omega'}$ belongs to $H(\Omega')$.

**Theorem II.3**: Let $y \in \mathbb{R}^N$ be given. Then $U_y : \mathbb{R}^N - \{y\} \to \mathbb{R}$ defined via

\[
U_y(x) = \begin{cases} 
- \log \|x - y\|, & \text{if } N = 2, \\
\|x - y\|^{2-N}, & \text{if } N \geq 3,
\end{cases}
\]

where $x \in \mathbb{R}^N - \{y\}$, is harmonic on $\mathbb{R}^N - \{y\}$. Moreover, if $f$ is harmonic on some annular region $A(y; r_1, r_2) := \{x \in \mathbb{R}^N \mid r_1 < \|x - y\| < r_2\}$, with $0 \leq r_1 < r_2 \leq \infty$ and depends only on $\|x - y\|$, then there are real numbers $\alpha$ and $\beta$ such that $f = \alpha U_y + \beta$.

**Proof**: Suppose that $f \in C^2(A(y; r_1, r_2))$ depends only on $\|x - y\|$, i.e. there exists a function $F \in C^2((r_1, r_2))$ such that for all $x \in A(y; r_1, r_2)$ it holds $f(x) = F(\|x - y\|)$. Put $r := \|x - y\|$. Then $\partial_i r = r^{-1}(x_i - y_i)$, so that $\partial_i f(x) = r^{-1}F'(r)(x_i - y_i)$ for $1 \leq i \leq N$ and

\[
\partial^2 f(x) = F''(r) \left( \frac{x_i - y_i}{r} \right)^2 + F'(r) \left( \frac{1}{r} - \frac{(x_i - y_i)^2}{r^3} \right).
\]

Thus $\Delta f(x) = F''(r) + r^{-1}(N - 1)F'(r)$. Hence, $f$ is harmonic on $A(y; r_1, r_2)$ if and only if $F$ solves the ordinary differential equation

\[
F''(r) + \frac{(N - 1)}{r}F'(r) = 0
\]

on the interval $(r_1, r_2)$. The only solutions are of the form

\[
F(r) = \begin{cases} 
- \alpha \log(r) + \beta, & \text{if } N = 2, \\
\alpha r^{2-N} + \beta, & \text{if } N \geq 3,
\end{cases}
\]

where $r \in (r_1, r_2)$. Thus, both assertions of the theorem are immediate. □

**Definition II.4**: We call the function $U_y$ as defined in Theorem II.3 the fundamental harmonic function for $\mathbb{R}^N$ with pole at $y$.

Potential theory is the study of harmonic functions in the same way as function theory is the study of holomorphic functions.
Remark II.5: The set of harmonic functions $H(\Omega)$ on some open set $\Omega$ is a vector space but no algebra (with respect to the pointwise multiplication). In fact, one has for two functions $f, g \in H(\Omega)$ that their pointwise product $f \cdot g$ is harmonic on $\Omega$ if and only if $\langle \operatorname{grad} f, \operatorname{grad} g \rangle \equiv 0$ on $\Omega$. Recall that $\operatorname{grad} f = (\partial_1 f, \ldots, \partial_N f)$ for every $f \in C^1(\Omega)$.

In the case $N = 2$, one observes a close relationship between potential theory and function theory. We identify $C \cong \mathbb{R}^2$ as abelian groups in the usual way, i.e. a complex number $z = x + iy$ is identified with the tuple $(x, y)$.

Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. We say that $f : \Omega \to \mathbb{C}$ is holomorphic on $\Omega$ if, for all $z_0 \in \Omega$, the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions $f : \Omega \to \mathbb{C}$. For $f \in \mathcal{O}(\Omega)$, in particular the partial derivatives do exist since for a point $z_0 = x_0 + iy_0 \in \Omega$ we have

$$\frac{\partial f}{\partial x}(z_0) = \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = f'(z_0)$$

$$\frac{\partial f}{\partial y}(z_0) = \lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{y - y_0} = if'(z_0). \quad (II.1)$$

For a partially differentiable function $f : \Omega \to \mathbb{C}$, we define the Pompeiu-Wirtinger derivatives by

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right), \quad \frac{\partial f}{\partial \bar{z}}(z_0) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0) \right).$$

For $f \in \mathcal{O}(\Omega)$, we see that for all $z_0 \in \Omega$ it holds $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$ and $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

Function theory teaches us that the holomorphic functions $\mathcal{O}(\Omega)$ on $\Omega$ are contained in the smooth functions $C^\infty(\Omega, \mathbb{C})$ on $\Omega$. Furthermore, a function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if $f$ belongs to $C^1(\Omega, \mathbb{C})$ and $\frac{\partial f}{\partial \bar{z}} \equiv 0$ on $\Omega$.

For $\Delta : C^2(\Omega, \mathbb{C}) \to C(\Omega, \mathbb{C})$, where $f = u + iv$ is sent to $\Delta u + i\Delta v$, we find that for all $f \in C^2(\Omega, \mathbb{C})$ it holds

$$\Delta f = 4 \frac{\partial^2 f}{\partial z \partial \bar{z}} = 4 \frac{\partial^2 f}{\partial z^2}. \quad (II.2)$$

Hence, for every holomorphic function $f : \Omega \to \mathbb{C}$, the Laplacian of $f$ vanishes on the whole of $\Omega$ and thus $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are both harmonic functions on $\Omega$. 

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Theorem II.6: Let \( \emptyset \neq \Omega \subseteq \mathbb{C} \) be a simply connected domain. Then, for every \( u \in H(\Omega) \), there is an \( f \in \mathcal{O}(\Omega) \) such that \( u = \text{Re}(f) \). We call the imaginary part \( v \) of \( f \), which is harmonic on \( \Omega \), the harmonic conjugate of \( u \).

Proof: Let \( u \) be harmonic on \( \Omega \). Then \( h := 2\partial_z u \in C^1(\Omega, \mathbb{C}) \) and, by Eq. (II.2), we find

\[
0 \equiv \Delta u = 4 \frac{\partial^2 u}{\partial \bar{z}\partial z} = 2 \frac{\partial h}{\partial \bar{z}}.
\]

Thus, \( h \) is holomorphic on \( \mathcal{O}(\Omega) \). Since \( \Omega \) is simply connected, there is a function \( f_0 \in \mathcal{O}(\Omega) \) with \( f'_0 = h \). Using Eq. (II.1) we find that

\[
\frac{\partial}{\partial x}(\text{Re}(f_0) - u) = \text{Re}\left(\frac{\partial f_0}{\partial x}\right) - \frac{\partial u}{\partial x} = \text{Re}(f'_0) - \text{Re}(h) = 0,
\]

\[
\frac{\partial}{\partial y}(\text{Re}(f_0) - u) = \text{Re}\left(\frac{\partial f_0}{\partial y}\right) - \frac{\partial u}{\partial y} = -\text{Im}(f'_0) + \text{Im}(h) = 0,
\]

and hence we conclude that \( \text{Re}(f_0) - u : \Omega \to \mathbb{R} \) is constant, say \( \text{Re}(f_0) - u \equiv c \) for some \( c \in \mathbb{R} \). Then, \( f := f_0 - c \) does the job. \( \square \)
Chapter III.

The Mean Value Property and its Consequences

Motivation III.1: Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open and let $f$ be holomorphic on $\Omega$. For a point $z_0 \in \Omega$ and a radius $r > 0$ such that $\text{cl}(D(z_0, r)) \subseteq \Omega$, we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - z_0} \, dz,$$

where the boundary is parametrised via $\gamma: [0, 2\pi] \to \mathbb{C}, \ t \mapsto z_0 + r \exp(it)$. Thus,

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(\gamma(t)) \frac{\gamma'(t)}{\gamma(t) - z_0} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \exp(it)) \, dt = \frac{1}{2\pi} \int_{S^1} f(z_0 + r \zeta) \, d\sigma^1(\zeta),$$

where $\sigma^1(\{\exp(it) | t \in (t_1, t_2)\}) = t_2 - t_1$, i.e. $f$ has the mean value property.

In particular,

$$f(z_0) \int_0^{r_0} r \, dr = \frac{1}{2} f(z_0) r_0^2 = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \exp(it)) \, r \, dt \, dr$$

and hence, $f(z_0) = (\pi r_0^2)^{-1} \int_{D(z_0, r_0)} f(z) \, d\lambda^2(z)$. This has many important consequences such as the famous maximus modulus principle. From Theorem II.6 it follows that every $u \in H(\Omega)$ has the mean value property. This is true not only in the case $N = 2$ but in full generality.

**Definition III.2:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and let $u$ be a continuous function on $\Omega$. We say that $u$ has the mean value property on $\Omega$ if for each $x_0 \in \Omega$ and
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every radius \( r > 0 \) with \( \text{cl}(B(x_0, r)) \subseteq \Omega \), where \( B(x_0, r) \) is the ball around \( x_0 \) with radius \( r \) with respect to the euclidean distance, it holds

\[
u(x_0) = \frac{1}{N\omega_N} \int_{\mathbb{S}^{N-1}} u(x_0 + r\zeta) \, d\sigma^{N-1}(\zeta) =: M(u; x_0, r),\]

or equivalently (see Exercise 2 (ii) on exercise sheet 1B)

\[
u(x_0) = \frac{1}{\omega_N r^N} \int_{B(x_0, r)} u(x) \, d\lambda_N(x) =: A(u; x_0, r).
\]

Here \( \lambda_N \) is the Lebesgue measure on \( \mathbb{R}^N \) and \( \sigma^{N-1} \) is the spherical measure on \( \mathbb{S}^{N-1} \) defined via \( \sigma^{N-1}(A) := N\lambda_N(\{ta \mid a \in A, t \in [0, 1]\}) \) for a Borel set \( A \in \mathcal{B}(\mathbb{S}^{N-1}) \). For the spherical measure it holds

\[
\int_{B(x_0, r)} f(x) \, d\lambda_N(x) = \int_0^r r^{-N} \int_{\mathbb{S}^{N-1}} f(x_0 + r\zeta) \, d\sigma^{N-1}(\zeta) \, dr. \quad (\text{III.1})
\]

Finally, \( \omega_N \lambda_N(B(0, 1)) = \Gamma(N/2 + 1)^{-1} \pi^{N/2} \) is the volume of the \( N \)-sphere. Note that \( \lambda_N(B(x_0, r)) = \omega_N r^N \) and \( \sigma^{N-1}(\mathbb{S}^{N-1}) = N\omega_N \).

We want to prove the following more general statement:

**Theorem III.3:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open. Then every \( u \in H(\Omega) \) has the mean value property.

The proof relies on the following fact:

**Theorem III.4 (Gauss’ Divergence Theorem):** If \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) is open and has a piecewise smooth boundary \( \partial \Omega \), then every function \( F = (f_1, \ldots, f_N) \) in \((C(\text{cl}(\Omega)) \cap C^1(\Omega))^N\) satisfies

\[
\int_{\Omega} (\text{div } F)(x) \, d\lambda_N(x) = \int_{\partial \Omega} \langle F(x), \nu(x) \rangle \, d\sigma_{\partial \Omega}(x).
\]

Here, \( \nu : \partial \Omega \to \mathbb{R}^N \) are the outer unit normal vectors to the surface \( \partial \Omega \), \( \sigma_{\partial \Omega} \) is the surface measure on \( \partial \Omega \) and \( \text{div } F \) is the divergence of \( R \), which is defined by \( \text{div } F(x) := \sum_{i=1}^N \partial_i F^i(x) \).

**Proof (of Theorem III.3):** We apply Theorem III.4 to \( B(x_0, r) \) and \( \text{grad } u \). We notice that \( \nu(x) = r^{-1}(x - x_0) \) and thus we get for \( u \in C^2(\Omega) \) that

\[
\int_{B(x_0, r)} \Delta u(x) \, d\lambda_N(x) = \int_{B(x_0, r)} (\text{div } \text{grad } u)(x) \, d\lambda_N(x).
\]

\[\]
\[ \hat{B}(x_0, r) = \int_{\partial B(x_0, r)} \langle \text{grad} u(x), r^{-1}(x - x_0) \rangle \, d\sigma_{\partial B(x_0, r)}(x). \]

Additionally note that \[ \int_{\partial B(x_0, r)} f(x) \, d\sigma_{\partial B(x_0, r)} = r^{N-1} \int_{S^{N-1}} f(x_0 + r\zeta) \, d\sigma^{N-1}(\zeta). \]

Using this fact, we can rephrase

\[ \int_{B(x_0, r)} (\Delta u)(x) \, d\lambda^N(x) = r^{N-1} \int_{S^{N-1}} \langle \text{grad} u(x_0 - r\zeta), \zeta \rangle \, d\sigma^{N-1}(\zeta) \]
\[ = r^{N-1} \int_{S^{N-1}} \frac{\partial u}{\partial r}(x_0 + r\zeta) \, d\sigma^{N-1}(\zeta) \]
\[ = r^{N-1} \frac{\partial}{\partial r} \int_{S^{N-1}} u(x_0 + r\zeta) \, d\sigma^{N-1}(\zeta), \]

which yields the formula

\[ rA(\Delta u; x_0, r) = N \frac{d}{dr} M(u; x_0, r). \quad (III.2) \]

Thus, if \( \Delta u \equiv 0 \), then \( M(u; x_0, \cdot) : (0, r_0) \to \mathbb{R} \) must be constant, where \( r_0 > 0 \) is such that \( B(x_0, r_0) \subseteq \Omega \). By part (i) of exercise 2 on exercise sheet 1B, it holds \( \lim_{r \downarrow 0} M(u; x_0, r) = u(x_0) \), hence for all \( r \in (0, r_0) \) it holds \( M(u; x_0, r) = u(x_0) \). \( \square \)

We will see that the local mean value property characterises harmonic functions. The following is the first step

**Theorem III.5:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open. Suppose that \( u \in C(\Omega) \) has the mean value property. Then \( u \) is smooth on \( \Omega \).

**Proof:** We consider a smooth function \( \phi \) on \( \mathbb{R} \) such that \( \phi(t) = 0 \) for all \( t \leq 0 \) and

\[ N\omega_M \int_0^1 t^{N-1} \phi(1 - t^2) \, dt = 1. \]

For each natural number \( N \), \( \phi_n(x) := n^N \phi(1 - n^2 \|x\|^2) \) defines a smooth function on \( \mathbb{R}^N \). Further, we put

\[ \Omega_n := \begin{cases} \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > 1/n\}, & \text{if } \Omega \neq \mathbb{R}^N, \\ \mathbb{R}^N, & \text{if } \Omega = \mathbb{R}^N \end{cases}. \]

Since \( \phi_n \) and all its derivations are supported on \( \text{cl}(B(0, 1/n)) \), we see that

\[ u_n(x) := \int_{\Omega_n} \phi_n(x - y) u(y) \, d\lambda^N(y) \]
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declares a smooth function \( u_n : \Omega_n \to \mathbb{R} \). Now, for \( x \in \Omega_n \), we have

\[
u_n(x) = \int_{B(x,1/n)} \phi_n(x - y)u(y) \, d\lambda^N(y)
\]

\[
= \int_0^{1/n} r^{N-1} \int_{S^{N-1}} \phi_n(r\zeta)u(x + r\zeta) \, d\sigma^{N-1}(\zeta) \, dr
\]

\[
= \int_0^{1/n} r^{N-1}n^N \phi(1 - u^2r^2) \int_{S^{N-1}} u(x + r\zeta) \, d\sigma^{N-1}(\zeta)
\]

\[
= u(x)N\omega_N \int_0^{1/n} r^{N-1}n^N \phi(1 - u^2r^2) \, dr = u(x).
\]

This tells us that \( u|_{\Omega_n} = u_n \) is smooth. Since \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \), it follows that \( u \) is a smooth function on \( \Omega \). \(\square\)

**Theorem III.6:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open and consider \( u \in C(\Omega) \). Then, the following are equivalent:

(i) \( u \) is harmonic on \( \Omega \),
(ii) \( u \) has the mean value property on \( \Omega \),
(iii) For all \( x_0 \in \Omega \) there is some \( r_0 > 0 \) such that \( B(x_0, r_0) \subseteq \Omega \) and \( u|_{B(x_0, r)} \) has the mean value property on \( B(x_0, r) \).

**Proof:** The implication “(i) \(\Rightarrow\) (ii)” is just **Theorem III.3** and the implication “(ii) \(\Rightarrow\) (iii)” is trivial.

For “(iii) \(\Rightarrow\) (i)”, we note that due to **Theorem III.5**, \( u|_{B(x_0, r_0)} \) is smooth. It suffices to show that \( \Delta u(x_0) = 0 \). From Eq. (III.1) and part (i) of exercise 2 of exercise sheet 1B, we infer that for every \( r \in (0, r_0) \) it holds

\[
N(M(u; x, r) - u(x)) = \int_0^r \rho A(\Delta u; x, \rho) \, d\rho,
\]

and thus, again with part (i) of exercise 2, we obtain

\[
N \lim_{r \to 0} \frac{1}{r^2} (M(u; x, r) - u(x)) = \lim_{r \to 0} \frac{1}{r^2} \int_0^r \rho A(\Delta u; x, \rho) \, d\rho = \frac{1}{2} \Delta u(x_0).
\]

The mean value property now enforces \( \Delta u(x_0) = 0 \). \(\square\)

**Corollary III.7:** For every open subset \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \), we have the inclusion \( H(\Omega) \subseteq C^\infty(\Omega) \). Moreover, if \( u \) is harmonic on \( \Omega \), then all partial derivatives of \( u \) belong to \( H(\Omega) \).
Proof: Take any \( u \in H(\Omega) \). By Theorem III.3, \( u \) has the mean value property on \( \Omega \) and thus, Theorem III.5 yields that \( u \in C^\infty(\Omega) \). The additional assertion follows by induction by using that \( \Delta \partial_k u = \partial_k \Delta u \) for all \( u \in C^\infty(\Omega) \). \( \square \)

The following result is a “harmonic counterpart” of the maximum modulus principle for holomorphic functions.

Theorem III.8: Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open and consider a holomorphic function \( u \) on \( \Omega \).

(i) If \( u \) attains a local maximum at some point \( x_0 \in \Omega \) (i.e. there is \( r_0 > 0 \) such that \( B(x_0, r_0) \subseteq \Omega \) and for all \( x \in B(x_0, r_0) \) it holds \( u(x) \leq u(x_0) \)), then \( u \) is constant in a neighbourhood of \( x_0 \). In fact, \( u \) is constant on \( B(x_0, r_0) \). The same condition holds for local minima.

(ii) If \( \Omega \) is connected and \( u \) attains a local extremum at some point \( x_0 \in \Omega \), then \( u \) is constant on \( \Omega \).

(iii) Denote by \( \partial^\infty(\Omega) \) be the boundary of \( \Omega \) in the one-point compactification \( \mathbb{R}^N \cup \{\infty\} \). Note that \( \infty \in \partial^\infty(\Omega) \) if and only if \( \Omega \) is unbounded. If \( u \) is harmonic on \( \Omega \), then it holds

\[
\min\{u(x) \mid x \in \partial^\infty(\Omega)\} = \min\{u(x) \mid x \in \Omega \cup \partial^\infty(\Omega)\}, \quad \text{(III.3)}
\]

\[
\max\{u(x) \mid x \in \partial^\infty(\Omega)\} = \max\{u(x) \mid x \in \Omega \cup \partial^\infty(\Omega)\}. \quad \text{(III.4)}
\]

Proof: (i) Since \( u \) has the mean value property, we see that for all \( 0 < r < r_0 \) it holds

\[
u(x_0) = A(u; x_0, r) = \frac{1}{\omega_{N+1}} \int_{B(x_0, r)} u(x) \, d\lambda^N(x) \leq u(x_0)
\]

where by continuity of \( u \), the last inequality is strict if

\[
\{x \in B(x_0, r) \mid u(x) = u(x_0)\} \subsetneq B(x_0, r).
\]

Thus \( u \) is constant on \( B(x_0, r) \) for each \( 0 < r < r_0 \) and so on \( B(x_0, r_0) \).

(ii) Consider the set \( \Omega_0 := \{x \in \Omega \mid u(x) = u(x_0)\} \). Note that \( \Omega_0 \) is non-empty, as \( x_0 \in \Omega_0 \). By continuity of \( u \), \( \Omega_0 \) is closed relative to \( \Omega \). Due to (i), \( \Omega_0 \) is also open. Hence, as \( \Omega \) is connected, if follows that \( \Omega_0 = \Omega \) and thus \( u \) is constant on the whole of \( \Omega \).

(iii) We prove Eq. (III.4) the proof of Eq. (III.3) is analogous. It suffices to show the inequality “\( \geq \)”, the inequality “\( \leq \)" is obvious. Take \( x_0 \in \Omega \cup \partial^\infty(\Omega) \) such that \( u(x_0) = \max\{u(x) \mid x \in \Omega \cup \partial^\infty(\Omega)\} \) and let \( \Omega_0 \) be any connected
component of \( \Omega \) for which \( x_0 \in \Omega_0 \cup \partial^\infty \Omega \). Now we have to distinguish cases: Firstly, if \( x_0 \in \Omega_0 \), by (ii) it follows that \( u \) is constant on \( \Omega_0 \) and thus, by continuity of \( u \), also on \( \Omega_0 \cup \partial^\infty \Omega_0 \). Hence
\[
\max\{u(x) \mid x \in \partial^\infty \Omega\} \geq \max\{u(x) \mid x \in \partial^\infty \Omega\} = u(x_0) = \max\{u(x) \mid x \in \Omega \cup \partial^\infty \Omega\}.
\]
Secondly, if \( x_0 \in \partial^\infty \Omega_0 \), then
\[
\max\{u(x) \mid x \in \partial^\infty \Omega\} \geq \max\{u(x) \mid x \in \partial^\infty \Omega\} \geq u(x_0) = \max\{u(x) \mid x \in \Omega \cup \partial^\infty \Omega\},
\]
which closes the proof.

\[\square\]

**Remark III.9:** Note that the proof of Theorem III.8 exclusively relies on the mean value property of \( u \). Due to Theorem III.6 this is however equivalent to \( u \) being harmonic. In fact, only the following condition is needed: For all \( x_0 \in \Omega \) is some radius \( r_0 = r_0(x) > 0 \) such that \( B(x_0, r_0) \) is contained in \( \Omega \) and for all \( 0 < r < r_0 \) it holds
\[
u(x) = A(u; x_0, r) \quad \text{or equivalently} \quad u(x) = M(u; x_0, r). \tag{III.5}
\]
This seems to be a weaker assumption, but in Chapter 4, we will prove the following result, which says that even Eq. (III.5) is equivalent to \( u \) being harmonic.

**Theorem III.10:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open and consider \( u \in C(\Omega) \). Then, the following are equivalent:

(i) \( u \) is harmonic on \( \Omega \),

(ii) \( u \) has the (local) mean value property,

(iii) \( u \) satisfies condition \( u(x) = A(u; x_0, r) \) \( \text{or equivalently} \quad u(x) = M(u; x_0, r). \tag{III.5} \)

The following is an analogue of Liouville’s Theorem for holomorphic functions.

**Theorem III.11:** Let \( u \) be harmonic on \( \mathbb{R}^N \) and bounded from below (or from above). Then \( u \) is constant.
Proof: Without loss of generality, we may suppose that for all $x \in \mathbb{R}^N$ it holds that $u(x) \geq 0$. Take $x, y \in \mathbb{R}^N$ and put $d := \|x - y\|$. Then, for every $r > 0$, the ball $B(x, r)$ is contained in $B(y, r + d)$ and hence by the mean value property and the positivity of $u$, we have

$$u(x) = \mathcal{A}(u; x, r)$$

$$\leq \frac{1}{\omega_{N+1}} \int_{B(y, r+d)} u(x) d\lambda^N(x)$$

$$= \left( \frac{r + d}{r} \right)^N \mathcal{A}(u; y, r + d) = \left( 1 + \frac{d}{r} \right)^N u(y).$$

If $r$ gets arbitrarily large, the right hand side of the above equation tends to $u(y)$. This shows that $u(x) \leq u(y)$. Since $x$ and $y$ were arbitrary, it follows that $u$ is constant. □
Chapter IV.

The Poisson Integral Formula for a Ball

The Poisson integral formula for balls can be seen as a “harmonic analogue” of Cauchy’s integral formula for holomorphic functions on disks, see Exercise 2 on exercise sheet 2B.

Definition IV.1 (Poisson kernel): Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. The function

$$K_{x_0,r}: B(x_0, r) \times \partial B(x_0, r) \to \mathbb{R}, \quad K_{x_0,r}(x,y) := \frac{1}{N\omega_N r} \frac{r^2 - \|x - x_0\|^2}{\|x - y\|^N}$$

is called the Poisson kernel of $B(x_0, r)$.

Lemma IV.2: Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given. For every $y \in \partial B(x_0, r)$, the function $B(x_0, r) \to \mathbb{R}, x \mapsto K_{x_0,r}(x,y)$ is harmonic on $B(x_0, r)$.

This assertion will be left as an exercise for the reader – namely in Exercise 1 on exercise sheet 2B.

Definition IV.3 (Poisson integral of a signed measure): Let $x_0 \in \mathbb{R}^N$ and let $r > 0$ be given. For a signed measure $\mu: \mathcal{B}(\partial B(x_0, r)) \to \mathbb{R}$ (i.e. a $\sigma$-additive function on the $\sigma$-algebra of Borel sets on $\partial B(x_0, r)$ with $\mu(\emptyset) = 0$ that does not take values $\pm \infty$), we call the function

$$I_{\mu, x_0,r}: B(x_0, r) \to \mathbb{R}, \quad I_{\mu, x_0,r}(x) := \int_{\partial B(x_0, r)} K_{x_0,r}(x,y) \, d\mu(y)$$

the Poisson integral of $\mu$. 21
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A better intuition for signed measures is instilled by the theorem on the so-called Hahn-Jordan decomposition:

**Theorem:** If $X$ is a topological space and $\mu : \mathcal{B}(X) \to \mathbb{R}$, then there are finite measures $\mu^+$ and disjoint $P, N \in \mathcal{B}(X)$ with $P \cup N = X$ such that $\mu^+(N) = 0 = \mu^-(P)$ and $\mu = \mu^+ - \mu^-$.  

In this situation, $\|\mu\| := \mu^+(X) + \mu^-(X)$ is called total variation of $\mu$.

If $f : \partial B(x_0, r) \to \mathbb{R}$ is integrable with respect to the surface measure $\sigma_{\partial B(x_0, r)}$, then we put $I_{f,x_0,r} := I_{\mu,x_0,r}$, where $\mu$ is the signed measure given by

$$d\mu(x) = f(y) \, d\sigma_{\partial B(x_0, r)}(y).$$

**Theorem IV.4:** Let $x_0 \in \mathbb{R}^N$ and $r > 0$ be given.

(i) If $\mu$ is a signed measure on $\partial B(x_0, r)$, then $I_{\mu,x_0,r}$ is harmonic on $B(x_0, r)$.

(ii) If $f : \partial B(x_0, r) \to \mathbb{R}$ is integrable with respect to the surface measure $\sigma_{\partial B(x_0, r)}$, then for $y \in \partial B(x_0, r)$, we have

$$\limsup_{B(x_0,r) \ni x \to y} I_{f,x_0,r}(x) \leq \limsup_{\partial B(x_0,r) \ni z \to y} f(z). \quad (IV.1)$$

Furthermore, if $f$ is continuous on $\partial B(x_0, r)$, then

$$\lim_{B(x_0,r) \ni x \to y} I_{f,x_0,r}(x) = f(y). \quad (IV.2)$$

**Proof:**

(i) Take any $\text{cl}(B(x, \rho)) \subseteq B(x_0, r)$. By Fubinis Theorem, we get

$$A(I_{\mu,x_0,r}; x, \rho) = \int_{\partial B(x_0,r)} A(K_{x_0,r}(-, y); x, \rho) \, d\mu(y)$$

$$= \int_{\partial B(x_0,r)} K_{x_0,r}(x, y) \, d\mu(y) = I_{\mu,x_0,r}$$

Hence, $I_{\mu,x_0,r}$ has the mean value property on $B(x_0, r)$. By Theorem III.6 we get that $I_{\mu,x_0,r}$ is harmonic on $B(x_0, r)$.

(ii) Firstly, for any constant $c \in \mathbb{R}$, we claim that $I_{c,x_0,r} \equiv c$. To see this, note that $I_{c,x_0,r}$ is harmonic on $B(x_0, r)$ and $I_{c,x_0,r}(x)$ depends only on $\|x - x_0\|$. Now, Theorem II.3 tells us that there are $\alpha, \beta \in \mathbb{R}$ such that for all $x \in B(x_0, r) - \{x_0\}$ it holds $I_{c,x_0,r} = \alpha U_{x_0} + \beta = A(x_0; 0, r)$. Since $\lim_{x \to x_0} I_{c,x_0,r}(x) = I_{c,x_0,r}(x_0) = c$, we must have $\alpha = 0$ and $\beta = c$ which yields the claim.
Secondly, suppose that for some \( c \in \mathbb{R} \), it holds \( \limsup_{\partial B(x_0,r) \ni z \to y} f(z) < c \).
If no such \( c \) exists, i.e. if the limes superior in question is infinity, there is nothing to prove. If now there is such a \( c \), there is some \( \delta > 0 \) such that
\[
\forall z \in B(y, 2\delta) \cap \partial B(x_0, r) : f(z) < c. \tag{IV.3}
\]
Then, for every \( x \in B(y, \delta) \cap B(x_0, r) \), it holds
\[
I_{f,x_0,r}(x) - c \equiv I_{f-c,x_0,r}(x) = h_1(x) - h_2(x),
\]
where
\[
h_1(x) = \int_{\partial B(x_0,r) - B(y,2\delta)} K_{x_0,r}(x, z)(f(z) - c) \, d\sigma_{\partial B(x_0,r)}(z),
\]
\[
h_2(x) = \int_{\partial B(x_0,r) \cap B(y,2\delta)} K_{x_0,r}(x, z)(f(z) - c) \, d\sigma_{\partial B(x_0,r)}(z);
\]
by Eq. (IV.3) we have \( h_2(x) < 0 \) and further
\[
|h_1(x)| \leq \max_{z \in \partial B(x_0,r) - B(y,2\delta)} K_{x_0,r}(x, z) \int_{\partial B(x_0,r)} (|f(z)| + |c|) \, d\sigma_{\partial B(x_0,r)}(z)
\]
\[
\leq \frac{1}{N \omega_N r} \frac{r^2 - \|x - x_0\|^2}{\delta^N} \int_{\partial B(x_0,r)} (|f(z)| + |c|) \, d\sigma_{\partial B(x_0,r)}(z)
\]
\[
\leq \frac{1}{N \omega_N r} \frac{r^2 - \|x - x_0\|^2}{\delta^N} \xrightarrow{x \to y} 0.
\]
Hence, \( \limsup_{B(x_0,r) \ni y \to y} I_{f,x_0,r}(x) \leq c \), which shows Eq. (IV.1).

(iii) For a continuous function \( f : \partial B(x_0, r) \to \mathbb{R} \), we apply Eq. (IV.1) to \( f \) and \(-f\), which yields the proof of Eq. (IV.2). \( \square \)

**Theorem IV.5 (Possions Integral Formula):** Let \( x_0 \in \mathbb{R}^N \) and let \( r > 0 \) be given. Then, every function \( u \in C(\operatorname{cl}(B(x_0, r))) \cap H(B(x_0, r)) \) satisfies for all \( x \in B(x_0, r) \) that
\[
u(x) = I_{u | \partial B(x_0,r),x_0,r}(x).
\]
**Proof:** By Theorem IV.4 (i), we have that \( v := u - I_{u,x_0,r} \) is harmonic on \( B(x_0, r) \) and due to Theorem IV.4 (ii), we have for all \( y \in \partial B(x_0, r) \) that
\[
\lim_{B(x_0,r) \ni x \to y} v(x) = 0.
\]
Thus, \( v \) extends to a function \( v \in H(B(x_0, r)) \cap C(\operatorname{cl}(B(x_0, r))) \) such that \( v|_{\partial B(x_0, r)} \equiv 0 \) and by Theorem III.8 (iii), it follows \( v \equiv 0 \), as desired. \( \square \)
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**Proof (of Theorem III.10):** The implication “(i) ⇒ (ii)” was shown in Theorem III.3 and the implication “(ii) ⇒ (iii)” is trivial.

For “(iii) ⇒ (i)”, take any $x_0 \in \Omega$ and $r > 0$ such that $\mathrm{cl}(B(x_0, r)) \subseteq \Omega$. We consider the function $v := u - I_{u,x_0,r}$. By Theorem IV.4 (i), $v$ is continuous on $B(x_0, r)$ and due to Theorem IV.4 (ii) we see that $v$ extends to a function $v \in C(\mathrm{cl}(B(x_0, r)))$ with $v|_{\partial B(x_0, r)} \equiv 0$.

Since $I_{u,x_0,r}$ is harmonic on $B(x_0, r)$ and $u$ satisfies Eq. (III.1) by assumption, also $v$ satisfies Eq. (III.1). Thus, Theorem III.8 (iii) can be applied (see Remark III.9) which gives $v \equiv 0$ on $\mathrm{cl}(B(x_0, r))$ and thus the restriction of $u$ to $B(x_0, r)$ equals $I_{u,x_0,r}$, which is harmonic on $B(x_0, r)$. In total, $u$ is harmonic on $\Omega$ as desired. □

**Theorem IV.6 (Harnack’s Inequalities):** Let $x_0 \in \mathbb{R}^N$ and let $r > 0$ be given. Suppose that $u$ is harmonic on $B(x_0, r)$ satisfying $u(x) \geq 0$ for all $x \in B(x_0, r)$. Then,

$$
\frac{(r - \|x - x_0\|)r^{N-2}}{(r + \|x - x_0\|)^{N-1}} u(x_0) \leq u(x) \leq \frac{(r + \|x - x_0\|)r^{N+2}}{(r - \|x - x_0\|)^{N-1}} u(x_0)
$$

holds for every point $x$ in $B(x_0, r)$.

**Proof:** Take any $0 < \rho < r$. Then, by Theorem IV.5, we have for all $x \in B(x_0, \rho)$ that $u(x) = I_{u,x_0,\rho}(x)$. Note that, for every $x$ in $B(x_0, \rho)$ and $y$ in $\partial B(x_0, \rho)$,

$$
K_{x_0,\rho}(x, y) = \frac{1}{\omega_N \rho^N} \rho^2 - \|x - x_0\|^2 \geq \frac{1}{\omega_N \rho^N} (\rho - \|x - x_0\|)(\rho + \|x - x_0\|) = \frac{1}{\omega_N \rho^N} \frac{\rho - \|x - x_0\|}{(\rho + \|x + x_0\|)^{N-1}}.
$$

Similarly, we deduce

$$
K_{x_0,\rho}(x, y) \leq \frac{1}{\omega_N \rho^N} \frac{\rho + \|x - x_0\|}{(\rho - \|x - x_0\|)^{N-1}}.
$$

Hence, by the mean value property of $u$, we find

$$
u(x) = \int_{\partial B(x_0, \rho)} K_{x_0,\rho}(x, y) u(y) d\sigma_{\partial B(x_0, \rho)}(y)
\geq \frac{1}{\omega_N \rho^N} \frac{\rho - \|x - x_0\|}{(\rho + \|x - x_0\|)^{N-1}} \int_{\partial B(x_0, \rho)} u(y) d\sigma_{\partial B(x_0, \rho)}(y)
$$

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\[
\frac{1}{\omega_N \rho N} \frac{\rho - \|x - x_0\|}{\rho + \|x - x_0\|^{N-1}} \rho^{N-1} N \omega_N u(x_0) = \frac{(\rho - \|x - x_0\|) \rho^{N-2}}{(\rho + \|x - x_0\|)^{N-1}} u(x_0).
\]

Again, with the same arguments, we see
\[
u(x) \leq \frac{(\rho + \|x - x_0\|) \rho^{N-2}}{(\rho - \|x - x_0\|)^{N-1}} u(x_0)
\]
Letting \(\rho \uparrow r\), we obtain the asserted bounds.

\[\square\]

**Corollary IV.7:** In the situation of **Theorem IV.6**, it holds that
\[
\|\nabla u(x_0)\| \leq \frac{N}{r} u(x_0).
\]

**Proof:** Let \(e \in \mathbb{R}^N\) with \(\|e\| = 1\) be given. Then
\[
f: (-r, r) \rightarrow \mathbb{R}, \quad t \mapsto u(x_0 - te)
\]
is well-defined and smooth as composition of smooth functions, whose derivative in zero is given by \(f'(0) = \langle \nabla u(x_0), e \rangle\). By **Theorem IV.6** we further have for \(t \in (0, r)\) that
\[
\frac{(r - t)r^{N-2}}{(r + t)^{N-1}} f(0) \leq f(t) \leq \frac{(r + t)r^{N-2}}{(r - t)^{N-1}} f(0).
\]
Hence,
\[
\frac{1}{t} \left( \frac{(r - t)r^{N-2}}{(r + t)^{N-1}} - 1 \right) f(0) \leq \frac{f(t) - f(0)}{t} \leq \frac{1}{t} \left( \frac{(r + t)r^{N-2}}{(r - t)^{N-1}} - 1 \right) f(0).
\]
Letting \(t \downarrow 0\), we get the bounds \(-N/r \leq f'(0) \leq N/r\). Therefore, we have the estimate \(|\langle \nabla u(x_0), e \rangle| = |f'(0)| \leq N/r\), from which the assertion follows. \(\square\)

**Remark IV.8:** The Dirichlet Problem on \(B(x_0, r)\) is to find, for a given function \(f \in C(\partial B(x_0, r))\), a function \(u\) which is harmonic on \(B(x_0, r)\) such that for all \(y \in \partial B(x_0, r)\) it holds
\[
\lim_{B(x_0, r) \ni x \rightarrow y} u(x) = f(y).
\]

**Theorem IV.4** tells us that \(u = I_{f, x_0, r}\) solves the Dirichlet Problem. Due to **Theorem III.8** this solution is in fact unique.
Chapter V.

Subharmonic functions

Subharmonic functions generalise harmonic functions; they are more flexible but have similar strong properties. This class is crucial for the study of harmonic functions.

**Definition V.1:** Let $X$ be a topological space.

A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is called **upper semicontinuous on $X$** or just **upper semicontinuous**, if $f^{-1}((-\infty, a))$ is open in $X$ for any real number $a$.

A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is called **lower semicontinuous on $X$** or just **lower semicontinuous**, if $-f$ is upper semicontinuous on $X$.

**Definition V.2:** Let $\emptyset \neq \Omega \subset \mathbb{R}^N$ be open. A function $s : \Omega \to (-\infty, +\infty)$ is called **subharmonic on $\Omega$** or briefly **subharmonic**, if $s$ is upper semicontinuous on $\Omega$, $s$ has the **subharmonic mean value property on $\Omega$**, i.e. for all $r > 0$ with $\text{cl}(B(x, r)) \subseteq \Omega$ it holds $s(x) \leq M(s; x, r)$ and $s \not\equiv -\infty$ on each connected component of $\Omega$. By $S(\Omega)$, we denote the set of all subharmonic functions on $\Omega$.

A function $u : \Omega \to (-\infty, +\infty]$ is called **superharmonic on $\Omega$** or briefly **superharmonic**, if $-u : \Omega \to [-\infty, +\infty)$ is subharmonic. Those functions have the **superharmonic mean value property**, i.e. for all $r > 0$ with $\text{cl}(B(x, r)) \subseteq \Omega$ it holds $u(x) \geq M(u; x, r)$. We denote by $U(\Omega)$ the set of all superharmonic functions on $\Omega$.

Note that the harmonic functions on $\Omega$ are precisely those that are both sub- and superharmonic on $\Omega$.

**Remark V.3:** (i) Let $f : X \to (-\infty, +\infty)$ be upper semicontinuous and let $K \subseteq X$ be compact. Then $\sup\{f(x) \mid x \in K\} < \infty$ and there exists $x_0 \in K$ such that $f(x_0) = \sup\{f(x) \mid x \in K\}$.
(ii) Let $s : \Omega \to [-\infty, +\infty)$ be upper semicontinuous. Decompose $s$ in its positive and negative parts $s^+ : \Omega \to [0, +\infty)$, $s^- : \Omega \to [0, +\infty]$ that are, for $x \in \Omega$, defined via $s^\pm(x) := \max\{\pm s(x), 0\}$. It holds $s = s^+ - s^-$ and $s^+$ is still upper semicontinuous. Thus, whenever $\text{cl}(B(x, r)) \subseteq \Omega$, then

$$\mathcal{M}(s; x, r) = \frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x, r)} s(\zeta) \, d\sigma(\zeta)$$

$$:= \frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x, r)} s^+(\zeta) \, d\sigma(\zeta) - \frac{1}{\omega_N N r^{N-1}} \int_{\partial B(x, r)} s^-(\zeta) \, d\sigma(\zeta)$$

where $\sigma = \sigma_{\partial B(x, r)}$. The integral over $s^+$ is bounded by $\max_{\zeta \in \partial B(x, r)} s^+(\zeta)$ which is finite and the integral of $s^-$ certainly is something in $[0, \infty]$. Thus, $\mathcal{M}(s; x, r)$ is a value in $[-\infty, \infty)$ and hence well-defined.

**Theorem V.4:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and let $s \in S(\Omega)$. Then, we have:

(i) For all $y \in \Omega$, it holds $\limsup_{x \to y} s(x) = s(y)$.
(ii) For all $r > 0$ with $\text{cl}(B(x, r)) \subseteq \Omega$ we have $s(x) \leq A(s; x, r)$.
(iii) The function $s$ is locally integrable, i.e. for every compact subset $K \subseteq \Omega$ it holds

$$\int_K |s(x)| \, d\lambda^N(x) < \infty.$$

**Proof:** (i) Since $s$ is upper semicontinuous, we have $\limsup_{x \to y} s(x) \leq s(y)$. If this inequality were strict, then we could find $r_0 > 0$ such that $B(y, r_0) \subseteq \Omega$ and for all $x \in B(y, r_0) - \{y\}$ it holds $s(x) < s(y)$. Thus, for each $0 < r < r_0$, by [Remark V.3](#),

$$\mathcal{M}(s; y, r) \leq \max\{s(\zeta) \mid \zeta \in \partial B(y, r)\} < s(y)$$

in contradiction to the subharmonic mean value property.

(ii) Like in Exercise 2 (ii) from sheet 1B, we get

$$r^N A(s; x, r) = N \int_{[0, r]} \rho^N \mathcal{M}(s; x, \rho) \, d\lambda^1(\rho) \quad (V.1)$$

and derive that $A(s; x, r) \geq s(x)$.

(iii) By Heine-Borel, it suffices to show that for each $x \in \Omega$, one finds $r > 0$ such that $\text{cl}(B(x, r)) \subseteq \Omega$ and Eq. (V.1) holds for $K = \text{cl}(B(x, r))$. In particular, without loss of generality, we may suppose that $\Omega$ is connected. Put

$$\Omega_0 := \left\{ x \in \Omega \mid \exists r > 0 : \text{cl}(B(x, r)) \subseteq \Omega, \int_{\text{cl}(B(x, r))} |s(x)| \, d\lambda^N(x) < \infty \right\}.$$
First of all, $\Omega_0$ is open. If we take $x \in \Omega_0$, there is some radius $r > 0$ such that $\text{cl}(B(x, r)) \subseteq \Omega$ and $\int_{\text{cl}(B(x,r))} |s(x)| d\lambda^N(x) < \infty$. We want to show that the whole of $B(x, r)$ is contained in $\Omega_0$. Take $x' \in B(x, r)$ and set $r' := r - |x' - x| > 0$. Then, $B(x', r') \subseteq B(x, r)$ and

$$\int_{\text{cl}(B(x', r'))} |s(x)| d\lambda^N(x) \leq \int_{\text{cl}(B(x, r))} |s(x)| d\lambda^N(x) < \infty,$$

i.e., $x' \in \Omega_0$, which shows $B(x, r) \subseteq \Omega_0$.

Secondly, $\Omega - \Omega_0$ is open and $s|_{\Omega-\Omega_0} \equiv -\infty$. To see this, let $x \in \Omega - \Omega_0$ be given. Choose $r > 0$ such that $\text{cl}(B(x, 2r)) \subseteq \Omega$. We want to show that $B(x, r)$ is contained in $\Omega - \Omega_0$ and that $s|_{B(x, r)} \equiv -\infty$. Let thus $x' \in B(x, r)$ and put $r' := r - |x' - x| \in (0, r)$. Then $B(x, r') \subseteq \Omega$ and hence, as $x \in \Omega - \Omega_0$, we must have $\int_{\text{cl}(B(x, r'))} |s(x)| d\lambda^N = \infty$. Since $\text{cl}(B(x, r')) \subseteq \text{cl}(B(x', r))$, we infer that also

$$\int_{\text{cl}(B(x', r))} |s(x)| d\lambda^N = \infty.$$  \hspace{1cm} (V.2)

However, as $\text{cl}(B(x', r)) \subseteq \text{cl}(B(x, 2r)) \subseteq \Omega$, we know that $s$ is bounded from above on $\text{cl}(B(x', r))$; hence, Eq. (V.2) gives that

$$\int_{\text{cl}(B(x', r))} s(x) d\lambda^N = -\infty$$

and so, $A(s; x', r) = -\infty$. Due to part (ii), it follows that $s(x') = -\infty$. Therefore, $s|_{B(x, r)} \equiv -\infty$ and consequently $B(x, r) \subseteq \Omega - \Omega_0$.

Finally, by Definition 5.2 (iii), we know that $s \not\equiv -\infty$, thus $\Omega_0 \neq \emptyset$. In summary, as $\Omega$ is connected, we get $\Omega = \Omega_0$. \hfill $\square$

**Corollary V.5:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open. For each $s \in S(\Omega)$, we have $\lambda^N(\{x \in \Omega \mid s(x) = -\infty\}) = 0$.

**Proof:** Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets $K_n$ of $\Omega$ such that $\Omega = \bigcup_{n=1}^{\infty} K_n$. Put $E := \{x \in \Omega \mid s(x) = -\infty\}$. Then $\lambda^N(K_n \cap E) = 0$ since $\int_{K_n} |s(x)| d\lambda^N(x) < \infty$ due to Theorem V.4. But because $E = \bigcup_{n=1}^{\infty} (K_n \cap E)$, $E$ is itself a null set. \hfill $\square$

**Example V.6:** (i) If $h$ is harmonic, then both $|h|$ and $h^2$ are subharmonic. This can be shown with help of the triangular inequality for integrals and the Cauchy Schwarz inequality.
(ii) It is a less obvious fact that \( s: \Omega \to \mathbb{R}, \ x \mapsto -\log(\text{dist}(x, \partial\Omega)) \) is subharmonic for every open set \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) where 
\[
\text{dist}(x, \partial\Omega) = \inf\{|x - y| \mid y \in \partial\Omega\}.
\]
is the distance of \( x \) to the set \( \Omega \).

The maximum principle for harmonic functions [Theorem III.8] generalises to subharmonic functions.

**Theorem V.7:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open and consider \( s \in S(\Omega) \).

(i) If \( s \) attains a local maximum at some point \( x_0 \in \Omega \), then \( s \) is constant on some neighbourhood of \( x_0 \).

(ii) If \( \Omega \) is connected and \( s \) attains a local maximum at some point \( x_0 \in \Omega \), then \( s \) is constant on \( \Omega \).

(iii) If \( u \in U(\Omega) \) is such that for all \( y \in \partial^\infty \Omega \) it holds
\[
\limsup_{x \to y} (s(x) - u(x)) \leq 0, \quad \text{(V.3)}
\]
then for all \( x \in \Omega \) it holds \( s(x) \leq u(x) \).

**Proof:** Statements (i) and (ii) can be shown like in the proof of [Theorem III.8]. To prove statement (iii), we may suppose that \( u \equiv 0 \), since \( s - u \) is subharmonic on \( \Omega \), and that \( \Omega \) is connected, since Eq. (V.3) holds for each connected component of \( \Omega \).

We define an upper semicontinuous function \( \bar{s}: \Omega \cup \partial^\infty \Omega \to [-\infty, +\infty) \), by 
\[
\bar{s}|_\Omega = s \quad \text{and for all } y \in \partial^\infty \Omega \text{ by } \bar{s}(y) := \limsup_{x \to y} s(x).
\]
By (Remark 5.3)(i), we find \( x_0 \in \Omega \cup \partial^\infty \Omega \) such that \( \bar{s}(x_0) = \sup\{s(x) \mid x \in \Omega \cup \partial^\infty \Omega\} \). If we had \( x_0 \in \Omega \), then \( s \) would have a constant positive value on \( \Omega \) due to (ii). By Eq. (V.3) this value must be less or equal to zero. Otherwise, if \( x_0 \in \partial^\infty \Omega \), then, for each \( x \in \Omega \), \( s(x) \leq \bar{s}(x_0) \leq 0 \) by Eq. (V.3). \( \square \)

Analogously, one has a minimum principle for superharmonic functions. Our next goal in the following is the characterisation of subharmonicity, see (Theorem 3.6), (Theorem 3.10) and (Theorem 4.5).

**Theorem V.8:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open and consider an upper semicontinuous function \( s: \Omega \to [-\infty, +\infty) \) which satisfies \( s \neq -\infty \) on each connected component of \( \Omega \). Then, the following are equivalent:

(i) The function \( s \) is superharmonic on \( \Omega \),
(ii) The function $s$ is bounded from above by $I_{s,x,r}$ on $B(x,r)$ whenever $\text{cl}(B(x,r)) \subseteq \Omega$.

(iii) For each $x \in \Omega$ such that $s(x) > -\infty$, we have

$$\limsup_{r \downarrow 0} \frac{1}{r^2} \mathcal{M}(s;x,r) - s(x) \geq 0.$$ 

(iv) For all $x \in \Omega$ there is some radius $r_0 > 0$ such that $B(x,r_0) \subseteq \Omega$ and for all $0 < r < r_0$ it holds $s(x) \leq \mathcal{M}(s;x,r)$.

(v) For all $x \in \Omega$ there is some radius $r_0 > 0$ such that $B(x,r_0) \subseteq \Omega$ and for all $0 < r < r_0$ it holds $s(x) \leq \mathcal{A}(s;x,r)$.

(vi) If $U$ is an open and bounded set such that $\text{cl}(U) \subseteq \Omega$ and if $h \in C(\text{cl}(U))$ is harmonic on $U$ and satisfies $s \leq h$ on $\partial U$, then $s \leq h$ on $U$.

The proof requires the following fact:

**Lemma V.9:** Let $\emptyset \neq X \subseteq \mathbb{R}^N$ be any subset and suppose $f : X \to [-\infty, +\infty)$ is upper semicontinuous on $X$ and bounded from above. Then there is a pointwise decreasing sequence $(f_n)_{n \in \mathbb{N}}$ in $C(\mathbb{R}^N)$ such that for all $x \in X$ it holds $f_n(x) \to f(x)$.

**Proof:** We will give a sketch of the proof of this statement. We extend $f$ to an upper semicontinuous function

$$\bar{f} : \mathbb{R}^N \to [-\infty, +\infty), \quad x \mapsto \begin{cases} f(x), & \text{if } x \in X, \\ \limsup_{y \to x} f(y), & \text{if } x \in \text{cl}(X) - X, \\ -\infty, & \text{else}. \end{cases}$$

If $\bar{f} \equiv -\infty$ on $\mathbb{R}^N$, then the sequence defined via $f_n \equiv -n$ does the trick. Otherwise, we define $f_n : \mathbb{R}^N \to \mathbb{R}$ via $f_n(x) := \sup_{y \in \mathbb{R}^N}(\bar{f}(y) - n\|x - y\|)$ and check that for all $x_1, x_2 \in \mathbb{R}^N$ it holds $|f_n(x_1) - f_n(x_2)| \leq n\|x_1 - x_2\|$. Thus, in this case, $f_n$ is continuous on $\mathbb{R}^N$ and one verifies that $f_n$ is pointwise decreasing and convergent to $\bar{f}$. 

**Proof (of Theorem V.8):** The implications “(i) $\Rightarrow$ (iv)” and “(iv) $\Rightarrow$ (iii)” are obvious.

For “(iii) $\Rightarrow$ (vi)” we consider the function $f : \mathbb{R}^N \to \mathbb{R}, \ y \mapsto \|y\|^2$ and put $a := \sup \{f(y) \mid y \in U\} < \infty$. For $\varepsilon > 0$ we put $u_\varepsilon := h - s - \varepsilon(f - a)$, which is a lower semicontinuous function on $\text{cl}(U)$ satisfying $u_\varepsilon \geq 0$ on $\partial U$. Now we set $b_\varepsilon := \inf \{u_\varepsilon(y) \mid y \in \text{cl}(U)\}$. 

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It holds $u_\varepsilon > b_\varepsilon$ on $U$, which can be seen as follows: Like in the proof of (Theorem III.6), we infer from (Eq. III.2)

$$\lim_{r \to 0} \frac{1}{r^2} (\mathcal{M}(f; y, r) - f(y)) = \frac{1}{2N} (\Delta f)(y) = 1$$

for all $y \in \mathbb{R}^N$, and thus, for every $y \in U$, it holds

$$\mathcal{M}(u_\varepsilon; y, r) = \mathcal{M}(h; y, r) - \mathcal{M}(s; y, r) - \varepsilon (\mathcal{M}(f; y, r) - a)$$

$$= h(y)\mathcal{M}(s; y, r) - \varepsilon (\mathcal{M}(f; y, r) - a),$$

and hence, for all sufficiently small $r > 0$, by (iii), we find

$$\frac{1}{r^2}(\mathcal{M}(u_\varepsilon; y, r) - u_\varepsilon(y)) = -\frac{1}{r^2}(\mathcal{M}(s; y, r) - s(y)) - \varepsilon \frac{1}{r^2}(\mathcal{M}(f; y, r) - f(y)) < 0,$$

i.e. $\mathcal{M}(u_\varepsilon; y, r) < u_\varepsilon(y)$. Therefore, if there was a point $y_0 \in U$ such that $u_\varepsilon(y_0) \leq b_\varepsilon$, then $b_\varepsilon \leq \inf\{u_\varepsilon(y) \mid y \in B(y_0, r)\} \leq \mathcal{M}(u_\varepsilon; y_0, r) < u_\varepsilon(y_0) \leq b_\varepsilon$, which is a contradiction. Hence, $u_\varepsilon > b_\varepsilon$ on all of $U$.

Due to (Remark V.3)(i), we find $y_0 \in \text{cl}(U)$ such that $u_\varepsilon(y_0) = b_\varepsilon$. By the preceeding claim, we must have $y_0 \in \partial U$ and hence $b_\varepsilon = u_\varepsilon(y_0) \geq 0$. Thus, $u_\varepsilon(y) \geq 0$ for all $y \in \text{cl}(U)$.

Letting $\varepsilon \downarrow 0$, we infer from the non-negativity of $u_\varepsilon$ on $\text{cl}(U)$ that $h \geq s$ on all of $U$.

“(iv) $\Rightarrow$ (ii)”: Consider $\text{cl}(B(x, r)) \subseteq \Omega$. By [Lemma V.9], there is a pointwise decreasing sequence $(f_n)_{n \in \mathbb{N}}$ in $C(\partial B(x, r))$ such that $f_n \to s$ on $\partial B(x, r)$. We define now define functions $h_n$ that are continuous on $\text{cl}(B(x, r))$ and harmonic on $B(x, r)$ via $h_n|_{B(x, r)} := I_{f_n, x, r}$ and $h_n|_{\partial B(x, r)} := f_n$. For the well-definedness of this sequence, check (Theorem IV.4)(ii). By assumption (vi), we have the estimate $s \leq h_n$ on $B(x, r)$. Now, the Beppo-Levi-Theorem implies that $h_n \to I_{s, x, r}$ on $B(x, r)$. Therefore, $s \leq I_{s, x, r}$ as desired.

For “(ii) $\Rightarrow$ (i)”: If we take $B(x, r)$ such that $\text{cl}(B(x, r))$ is contained in $\Omega$, then (ii) gives that $s(x) \leq I_{s, x, r}(x) = \mathcal{M}(s; x, r)$, which shows that $s$ has the subharmonic mean value property on $\Omega$, i.e. $s \in S(\Omega)$.

So far, we have shown (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (vi). Now we have just have to weave in (v). “(i) $\Rightarrow$ (v)” follows from (Theorem V.4)(ii). Now for “(v) $\Rightarrow$ (iii)”:

For $0 < r < r_0$, we define from [Eq. (V.1)] that

$$0 \leq r^N (\mathcal{M}(s; x, r) - s(x)) = N \int_{[0,r]} \rho^N (\mathcal{M}(s; x, \rho) - \rho(x)) d\lambda^1(\rho)$$
Thus, if we had that \( \limsup_{r \downarrow 0} \frac{1}{r^2}(\mathcal{M}(s; x, r) - s(x)) < 0 \), then there were some \( a > 0 \) positive such that

\[
\frac{1}{a^2}(\mathcal{M}(s; x, r) - s(x)) \leq -a < 0
\]

for all sufficiently small values of \( r \) and hence \( 0 \leq -Na \int_{(0,r)} \rho^{N+2} d\lambda^1(\rho) < 0 \), a contradiction. Therefore, (iii) must hold. \( \square \)

The characterisation (vi) of subharmonic functions explains the name “subharmonic”.

**Corollary V.10:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be given. If \( s_1, s_2 \) are subharmonic on \( \Omega \) and \( s_1 = s_2 \lambda^N \)-almost everywhere on \( \Omega \), i.e. \( \lambda^N\{x \in \Omega \mid s_1(x) \neq s_2(x)\} = 0 \), then \( s_1 = s_2 \) on \( \Omega \).

**Proof:** First, we claim that for a subharmonic function \( s \) on \( \Omega \) and every \( x \in \Omega \), it holds \( \lim_{r \downarrow 0} A(s; x, r) = s(x) \).

This can be seen as follow: By assertion (v) of Theorem V.8, we know that \( \liminf_{r \downarrow 0} A(s; x, r) \geq s(x) \) and by (Theorem V.4)(i), we know that

\[
\limsup_{r \downarrow 0} A(s; x, r) \leq \lim_{y \to x} s(y) = s(x).
\]

Thus, \( \lim_{r \downarrow 0} A(s; x, r) \) exists and equals \( s(x) \).

Secondly, for two subharmonic functions \( s_1, s_2 \) on \( \Omega \) with \( s_1 = s_2 \lambda^N \)-almost everywhere it follows for every \( x \in \Omega \) and all \( 0 < r < r_0 \) that

\[
A(s_1; x, r) = A(s_2; x, r)
\]

whenever \( B(x, r_0) \subseteq \Omega \). By our first claim, this show that \( s_1(x) = s_2(x) \). \( \square \)

**Corollary V.11:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be an open set and let \( s \in C^2(\Omega) \). Then \( s \) is subharmonic if and only if \( \Delta s \geq 0 \) on \( \Omega \).

**Proof:** Since for all \( x \in \Omega \) it holds

\[
\lim_{r \downarrow 0} \frac{1}{r^2}(\mathcal{M}(s; x, r) - s(x)) = \frac{1}{2N} (\Delta s)(x),
\]

the asserted equivalence is precisely the equivalence of assertions (i) and (iii) of Theorem V.8. \( \square \)
In view of this result Corollary V.11, it is desirable to know how to approximate subharmonic functions by smooth functions. For this purpose, we have the following:

**Theorem V.12:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and let $s$ be a subharmonic function on $\Omega$. Furthermore, let $\emptyset \neq U \subseteq \mathbb{R}^N$ be open and bounded with $\text{cl}(U) \subseteq \Omega$. Then, there is a sequence $(s_n)_{n \in \mathbb{N}}$ in $S(\Omega) \cap C^\infty(U)$ which is pointwise decreasing on $U$ with $s(x) = \lim_{n \to \infty} s_n(x)$ for all $x \in U$.

We record an important consequence:

**Theorem V.13:** Let $\emptyset \neq \Omega_1, \Omega_2 \subseteq \mathbb{C}$ be domains and suppose that $f \in \mathcal{O}(\Omega_1)$ is non-constant with $f(\Omega_1) \subseteq \Omega_2$. Then, for a subharmonic function $s$ on $\Omega_2$ it holds that $s \circ f$ is subharmonic on $\Omega_1$.

**Proof:** Take any point $z_0 \in \Omega_1$ and choose a radius $r_2 > 0$ such that the open set $U_2 := D(f(z_0), r)$ satisfies $\text{cl}(U_2) \subseteq \Omega_2$. Furthermore, take any radius $r_1 > 0$ such that $U_1 := D(z_0, r_1)$ satisfies $f^{-1}(U_2)$. It suffices to prove that $s \circ f|_{U_1}$ is subharmonic on $U_1$, since being subharmonic is a local property due to Theorem V.8 and since $z_0 \in \Omega_1$ was arbitrary.

By Theorem V.12 we find a sequence $(s_n)_{n \in \mathbb{N}}$ in of smooth subharmonic functions on $U_2$ which is pointwise decreasing on $U_2$ with limit $s|_{U_2}$. Therefore, $(s_n \circ f|_{U_1})_{n \in \mathbb{N}}$ is pointwise decreasing on $U_1$ with limit $s \circ f|_{U_1}$. Furthermore, $s_n \circ f$ is itself smooth on $U_1$. Because for all $z \in U_1$ it holds

$$\Delta (s_n \circ f)(z) = (\Delta s_n)(f(z))|f'(z)| \geq 0,$$

we know that $s_n \circ f$ is in fact subharmonic on $U_1$ (see Corollary V.11). By the Open Mapping Theorem (refer to Satz 13.5 of the lecture notes “Funtionentheorie I” from summer 2017), we know that $f(U_1)$ is open and thus, by (Theorem V.4)(iii), $s \not\equiv -\infty$ on $f(U_1)$ and so $s \circ f|_{U_1} \not\equiv -\infty$ on $U_1$. From exercise 2, assignment 3B, it follows that $s \circ f|_{U_1}$ is subharmonic as desired. \qed

The proof of Theorem V.12 uses convolution with smooth functions like in (Theorem III.5). We skip the details. We note however that the proof uses the following fact, which is a consequence of Fatou’s Lemma and Fubini’s Theorem:

**Theorem V.14:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and connected. Furthermore, let $(Y, \mu)$ be a $\sigma$-finite measure space. Suppose that $f : \Omega \times Y \to (-\infty, \infty]$ is measurable such that for all $y \in Y$ the function $f(\cdot, y)$ is superhamonic on $\Omega$ and
there is $\mu$-integrable functions $g: Y \to \mathbb{R}$ which satisfies for all $(x,y) \in \Omega \times Y$ that $f(x,y) \geq g(y)$. Then

$$u(x) := \int_Y f(x,y) \, d\mu(y)$$

declares a function $u: \Omega \to (-\infty, +\infty]$ which is either constantly $+\infty$ on $\Omega$ or superharmonic on $\Omega$.

**Remark V.15:** Subharmonic functions have some remarkable applications in functional analysis.

Let $(A, \| \cdot \|)$ be a unital complex Banach algebra. For $a \in A$, the set

$$\sigma(a) := \{ z \in \mathbb{C} \mid a - z \text{id}_A \text{ not invertible in } A \}$$

is called the *spectrum of $A$* and the real number $r(a) := \max\{|z| \mid z \in \sigma(a)\}$ is called the *spectral radius of $a$*. One can show that the following identity holds:

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}.$$

Vesentini’s Theorem says that, for every holomorphic function $f: \Omega \to A$ on a non-empty open set $\Omega \subseteq \mathbb{C}$, the function

$$s: \Omega \longrightarrow [-\infty, +\infty), \quad z \longmapsto \log r(f(z))$$

is subharmonic, provided that $s \neq -\infty$ on each connected component of $\Omega$.

Using this remarkable fact, one can prove Johnson’s Theorem which says that a surjective homomorphism $\Theta: A_1 \to A_2$ between complex Banach algebras $A_1$ and $A_2$ is automatically continuous if $A_2$ is semisimple (i.e. the intersection of all maximal right ideals in $A_2$ is $\{0\}$).
Chapter VI.

Riesz Measure

In Corollary V.11 we have seen that subharmonic $C^2$-functions have a non-negative Laplacian. Here, we generalise this observation to arbitrary subharmonic functions: The Laplacian of a subharmonic function, if understood in a distributional sense, yields then the Riesz Measure.

For a non-empty open subset $\emptyset \neq \Omega \subseteq \mathbb{R}^N$, we denote by $C_c(\Omega)$ the space of compactly supported continuous functions on $\Omega$, i.e. of functions $f: \Omega \to \mathbb{R}$ such that $\text{supp } f := \text{cl}_\Omega(\{x \in \Omega \mid f(x) \neq 0\})$ is compact. By $C_c^\infty(\Omega)$ we denote the space of smooth compactly supported continuous functions on $\Omega$.

**Definition VI.1:** Let $u: \Omega \to [-\infty, +\infty]$ be locally integrable on $\Omega$. The **distributional Laplacian of** $u$ is the linear functional

$$L_u: C^\infty_c(\Omega) \longrightarrow \mathbb{R}, \quad L_u(f) := \int_\Omega u(x)(\Delta f)(x) d\lambda^N(x).$$

**Theorem VI.2:** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ open.

(i) If $u \in C^2(\Omega)$, then for all $f \in C^\infty_c(\Omega)$ it holds

$$L_u(f) = \int_\Omega (\Delta u)(x)f(x) d\lambda^N(x).$$

(ii) If $h$ is harmonic on $\Omega$, then $L_h \equiv 0$ on $C^\infty_c(\Omega)$.

(iii) If $s$ is subharmonic on $\Omega$, then $L_s$ is a positive linear functional on $C^\infty_c(\Omega)$.

**Proof:**

(i) This follows from Greens Identity (see exercise 1, assignment 4B).

(ii) This is an immediate consequence of (i).
Chapter VI. Riesz Measure

(iii) Take any \( f \in C_c^\infty(\Omega) \) that is pointwise non-negative and let \( \emptyset \neq U \subseteq \mathbb{R}^N \) be open such that \( \text{supp } f \subseteq U \). By Theorem V.12 there is a sequence \( (s_n)_{n \in \mathbb{N}} \) of smooth subharmonic functions on \( U \) which is pointwise decreasing on \( U \) and convergent to \( s \). By Corollary V.11 we know that for each natural number, \( \Delta s_n \) is pointwise non-negative on \( U \) and thus, by (i), we get that

\[
\int_U s_n(x)(\Delta f)(x) \, d\lambda^N(x) = \int_U f(x)(\Delta s_n)(x) \, d\lambda^N(x) \geq 0.
\]

By the monotone convergence of \( (s_n(\Delta f)^\pm)_{n \in \mathbb{N}} \) on \( U \), it follows that

\[
L_s(f) = \int_U s(x)(\Delta f)(x) \, d\lambda^N(x) = \lim_{n \to \infty} \int_U s_n(x)(\Delta f)(x) \, d\lambda^N(x) \geq 0
\]

which we wanted to see. \( \square \)

Reminder (Radon Measure): A measure \( \mu \) on an open set \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) is called a Radon measure, if for all compact subsets \( K \subseteq \Omega \) it holds \( \mu(K) < \infty \).

Theorem VI.3: Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be an open subset and let \( s \) be a subharmonic function on \( \Omega \). Then there is a unique Radon measure \( \mu_s \) on \( \Omega \), called the Riesz measure associated to \( s \), such that for all \( f \in C_c^\infty(\Omega) \) it holds

\[
L_s(f) = \int_\Omega f(x) \, d\mu_s(x).
\]

Proof: We show the assertion in three steps.

(1) The distributional Laplacian \( L_s \) extends to a positive linear functional \( \hat{L}_s : C_c(\Omega) \to \mathbb{R} \).

By convolution with smooth functions, one can show that for all functions \( f \in C_c(\Omega) \) there is a sequence of smooth compactly supported functions \( (f_n)_{n \in \mathbb{N}} \) on \( \Omega \) such that

\[
\sup\{|f(x) - f_n(x)| \mid x \in \Omega\} \longrightarrow 0 \quad (n \to \infty).
\]  

(VI.1)

Therefore, \( (L_S(f_n))_{n \in \mathbb{N}} \) is a Cauchy sequence. One can show that the assignment \( \hat{L}_s(f) := \lim_{n \to \infty} L_s(f_n) \) is independent of the approximating sequence and thus we obtain a well-defined linear extension of \( L_s \) to \( C_c(\Omega) \).
(2) By Riesz Representation Theorem, there is a (unique) Radon measure \( \mu_s \) on \( \Omega \) such that for all \( f \in C_c(\Omega) \) it holds
\[
\hat{L}_s(f) = \int_{\Omega} f(x) \, d\mu_s(x).
\]

(3) The obtained Radon measure is unique in the sense that we desire.
Suppose \( \mu_1, \mu_2 \) were Radon measures on \( \Omega \) such that for all \( f \in C_\infty^c(\Omega) \) it holds
\[
\int_{\Omega} f(x) \, d\mu_1(x) = \int_{\Omega} f(x) \, d\mu_2(x),
\]
then we had \( \mu_1 = \mu_2 \), because by Eq. (VI.1), we find that Eq. (VI.2) extends to holds for all \( f \in C_c(\Omega) \). The uniqueness part of the Riesz Representation Theorem hence yields that \( \mu_1 = \mu_2 \). □

For a superharmonic function \( u \) on the open non-empty subset \( \Omega \) of \( \mathbb{R}^N \), we define the Riesz measure \( \mu_u \) associated to \( u \) as the Riesz measure associated to \( -u \), which is subharmonic on \( \Omega \).

**Theorem VI.4:** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^N \) be open.

(i) The fundamental harmonic function \( U_y \) for \( \mathbb{R}^N \) with pole at \( y \) \( \in \mathbb{R}^N \) (see Definition II.4) extends by \( U_y(y) := \infty \) to a superharmonic function. Its Riesz measure is given by \( \mu_{U_y} = a_N \delta_y \), where \( a_N = \max\{1, N - 2\} N\omega_N \) and \( \delta_y \) is the Dirac measure with atom \( y \).

(ii) Let \( \mu \) be a finite measure on \( \mathbb{R}^N \) whose support
\[
\text{supp } \mu := \{ x \in \mathbb{R}^N \mid \forall \varepsilon > 0 : \mu(B(x, \varepsilon)) > 0 \}
\]
is compact. Then the potential \( \phi_\mu \) associated to \( \mu \), which is defined by
\[
\phi_\mu(x) := \int_{\mathbb{R}^N} U_y(x) \, d\mu(y),
\]
is itself superharmonic on \( \mathbb{R}^N \) and harmonic on \( \mathbb{R}^N - \text{supp } \mu \). The Riesz measure of \( \phi_\mu \) is given by \( \mu_{\phi_\mu} = a_N \mu \).

**Proof:** (i) By (Theorem II.3), \( U_y \) is harmonic on \( \mathbb{R}^N - \{y\} \). Using the implication “(vi) \( \Rightarrow \) (i)” of [Theorem V.8][1] it follows that \( U_y \) is superharmonic on \( \mathbb{R}^N \). For \( \mu_{U_y} = a_N \delta_y \), it suffices to show that for all compactly supported smooth functions \( f \) on \( \mathbb{R}^N \) it holds that \( L_{U_y}(f) = a_N f(y) \). Let thus \( f \in C_c^\infty(\mathbb{R}) \)
and let \( r > 0 \) be such that \( \text{supp } f \subseteq B(y, r) \). By Green’s Identity (see Exercise 1 of assignment 4B), we get for \( \varepsilon > 0 \) that

\[
\int_{A(y;\varepsilon,r)} U_y(x)(\Delta f)(x) \, d\lambda^N(x) = -\int_{\partial B(y;\varepsilon)} U_y(x)\langle \text{grad } f(x), \nu(x) \rangle \, d\sigma(x)
- \int_{\partial B(y;\varepsilon)} f(x)\langle \text{grad } U_y(x), \nu(x) \rangle \, d\sigma(x)
\]

with \( \sigma := \sigma_{\partial B(y;\varepsilon)} \) and \( \nu(x) = (x - y)/\|x - y\| \). Denote the first integral on the right-hand side by \( I^1_\varepsilon \) and the second integral on the right-hand side by \( I^2_\varepsilon \).

Since for all \( x \in \mathbb{R}^N - \{y\} \) it holds that

\[
\text{grad } U_y(x) = -\max\{1, N - 2\} \frac{\nu(x)}{\|x - y\|^{N-1}},
\]

we infer that, as \( \varepsilon \downarrow 0 \), \( I^2_\varepsilon = -a_N/\varepsilon^{N-1} \int_{\partial B(y;\varepsilon)} f(x) \, d\sigma(x) \to -a_N f(y) \). Since for all \( x \in \partial B(y, \varepsilon) \) it holds that

\[
U_y(x) = \begin{cases} 
-\log \varepsilon, & \text{if } N = 2, \\
\varepsilon^{2-N}, & \text{if } N \geq 3,
\end{cases}
\]

we infer (with the help of the Cauchy-Schwarz-inequality applied to the integrand)

\[
|I^1_\varepsilon| \leq \max_{x \in \partial B(y;\varepsilon)} \|\text{grad } f(x)\| \int_{\partial B(y;\varepsilon)} U_y(x) \, d\sigma(x) \to 0 \quad \text{as } \varepsilon \downarrow 0,
\]

and hence \( I^1_\varepsilon \to 0 \) as \( \varepsilon \downarrow 0 \). In summary, we get that

\[
L_{-U_y}(f) = -\lim_{\varepsilon \downarrow 0} \int_{A(y;\varepsilon,r)} U_y(x)(\Delta f)(x) \, d\lambda^N(x) = a_N f(y).
\]

(ii) Due to [Theorem V.8](#) it suffices to check for all \( x_0 \in \mathbb{R}^N \) and \( r > 0 \) with \( B(x_0, r) \cap \mathbb{R}^N - \text{supp } \mu \neq \emptyset \) that \( \Phi_\mu|_{B(x_0,r)} \) is superharmonic on \( B(x_0, r) \). For every such ball \( B(x_0, r) \), we apply (Theorem V.14) to the function

\[
f: B(x_0, r) \times \Omega \to (-\infty, +\infty], \quad f(x, y) := U_y(x).
\]

Because we know that \( \Phi_\mu(x) < \infty \) for each \( x \in \mathbb{R}^N - \text{supp } \mu \), this gives that \( \Phi_\mu|_{B(x_0,r)} \) is superharmonic on \( B(x_0, r) \). If \( B(x_0, r) \) is contained in \( \mathbb{R}^N - \text{supp } \mu \), then we can apply this argument to \(-U_y\), which yields that \( \Phi_\mu|_{B(x_0,r)} \) is in fact...
harmonic on $B(x_0, r)$. In order to prove $\mu_\Phi = a_N \mu$, we must show that for all compactly supported smooth functions $f$ on $\mathbb{R}^N$ it holds

$$L_{-\Phi}(f) = a_N \int_{\mathbb{R}^N} f(y) \, d\lambda^N(y).$$

This follows with the help of Fubini’s Theorem from (i):

$$L_{-\Phi}(f) = -\int_{\mathbb{R}^N} \Phi(x)(\Delta f)(x) \, d\lambda^N(x)$$

$$= -\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} U_y(x) \, d\mu(y) \right) (\Delta f)(x) \, d\lambda^N(x)$$

$$= \int_{\mathbb{R}^N} \left( -\int_{\mathbb{R}^N} U_y(x)(\Delta f)(x) \, d\lambda^N(x) \right) \, d\mu(y) = \int_{\mathbb{R}^N} a_N f(y) \, d\mu(y),$$

which we wanted to show. \qed

**Remark VI.5:** Let $\rho: \mathbb{R}^N \to \mathbb{R}$ be in $C^k_c(\mathbb{R}^N) = C_c(\mathbb{R}^N) \cap C^k(\mathbb{R}^N)$ for some $k \geq 1$. On $\mathbb{R}^N$ one can declare a function $\Phi$ via

$$\Phi(x) := \int_{\mathbb{R}^N} U_y(x) \rho(y) \, d\lambda^N(y),$$

which in fact belongs to $C^{k+1}(\mathbb{R}^N)$.

Furthermore, this function $\Phi$ solves *Poisson's equation* $\Delta \Phi = -a_N \rho$ on $\mathbb{R}^N$. Indeed, by Theorem VI.2(i), we have for all $f \in C^\infty_c(\mathbb{R}^N)$ that

$$L_{\Phi}(f) = \int_{\mathbb{R}^N} f(x) \Delta \Phi(x) \, d\lambda^N(x),$$

whereas, for the measures $\mu^\pm$ defined by $d\mu^\pm(y) := \rho^\pm(y) \, d\lambda^N(y)$, Theorem VI.4(ii) yields that

$$L_{\Phi}(f) = L_{\Phi^+}(f) - L_{\Phi^-}(f)$$

$$= -a_N \int_{\mathbb{R}^N} f(x) \, d\mu^+(x) + a_N \int_{\mathbb{R}^N} f(x) \, d\mu^-(x)$$

$$= -a_N \int_{\mathbb{R}^N} f(x) \rho(x) \, d\lambda^N(x),$$

where the first step relies on the decomposition $\Phi = \Phi^+ - \Phi^-$ which is valid thanks to $\rho = \rho^+ - \rho^-$. Hence, in summary, for all $f \in C^\infty_c(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} f(x) \left( \Delta \Phi(x) + a_N \rho(x) \right) \, d\lambda^N(x) = 0,$$

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which implies $\Delta \Phi(x) = -a_N \rho(x)$ for all $x \in \mathbb{R}^N$.

In particular, for $N = 3$, since $a_3 = 4\pi$, we get $\Delta \Phi = -4\pi \rho$. In the notation of Chapter 1, we have $\text{grad } \Phi = -4\pi \varepsilon_0 \vec{E}$, so that the latter identity yields $\text{div } \vec{E} = \rho/\varepsilon_0$, which is known as the differential form of Gauss’ Law from material on classical electrodynamics.
Chapter VII.

Logarithmic Potentials

From now on, we draw our attention to the case $N = 2$. We therefore identify $\mathbb{R}^2$ and $\mathbb{C}$. If $\mu$ is a finite (Borel) measure on $\mathbb{C}$ with compact support $K := \text{supp} \mu$, then we refer to the potential

$$
\Phi_{\mu} : \mathbb{C} \rightarrow (-\infty, +\infty], \quad z \mapsto -\int_{K} \log|z - w| \, d\mu(w),
$$

which was defined in Theorem VI.4(ii), as the logarithmic potential associated to $\mu$. The logarithmic potential associated to $\mu$ is superharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} - K$.

**Theorem VII.1:** Let $\mu$ be a finite measure on $\mathbb{C}$ with compact support $K$.

(i) $\Phi_{\mu}(z) = -\mu(\mathbb{C}) \log |z| + \log |z| + O(1/|z|)$ as $z \to \infty$,

(ii) Let $w_0 \in K$. Then

$$
\limsup_{z \to w_0} \Phi_{\mu}(z) = \limsup_{K \ni w \to w_0} \Phi_{\mu}(w).
$$

Furthermore, if $\lim_{K \ni w \to w_0} \Phi_{\mu}(w) = \Phi_{\mu}(w_0)$, then it holds

$$
\lim_{z \to w_0} \Phi_{\mu}(z) = \Phi_{\mu}(w_0).
$$

(iii) If there is a real bound $M$ such that for all $w \in K$ it holds $\phi_{\mu}(w) \leq M$, then it even holds $\phi_{\mu}(z) \leq M$ for all $z \in \mathbb{C}$.

**Definition VII.2:** Let $\mu$ be a finite measure on $\mathbb{C}$ with compact support $K$. We call

$$
I(\mu) := \int_{K} \Phi_{\mu}(z) \, d\mu(z) = -\int_{K} \int_{K} \log|z - w| \, d\mu(z) \, d\mu(w)
$$

the energy of $\mu$.  

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Chapter VII. Logarithmic Potentials

Definition VII.3:

(i) A subset $E \subseteq \mathbb{C}$ is called polar if $I(\mu) = \infty$ for every finite measure $0 \neq \mu$ with compact support $\text{supp} \mu \subseteq E$.

(ii) A property is said to hold nearly everywhere on $S \subseteq \mathbb{C}$ if it holds on $S - E$ for some Borel polar set $E$.

Theorem VII.4:

(i) If $\mu$ is a finite measure on $\mathbb{C}$ with compact support satisfying $I(\mu) < \infty$, then $\mu(E) = 0$ for every Borel polar set.

(ii) Every Borel polar set $E \subseteq \mathbb{C}$ satisfies $\lambda^2(E) = 0$. In particular we find that “nearly everywhere” is stronger than “almost everywhere”\footnote{One direction is clear from the above, the other direction can be shown but requires work.}

(iii) A countable union of Borel polar sets is polar.

Definition VII.5: Let $K \subseteq \mathbb{C}$ be compact. We denote by $P(K)$ the set of all probability measures on $K$ (i.e. measure $\mu$: $\mathcal{B}(K) \to [0,1]$ with $\mu(K) = 1$). If there exists $\nu \in P(K)$ such that $I(\nu) = \inf_{\mu \in P(K)} I(\mu)$, then $\nu$ is called an equilibrium measure for $K$.

Theorem VII.6: Every compact set $K \subseteq \mathbb{C}$ has an equilibrium measure.

Proof: We give a sketch of the proof. To achieve this, we list some facts:

1. If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in $P(K)$ which is weak *-convergent to some $\mu \in B(K)$, i.e. for all $f \in C(K)$ it holds $\int_K f(z) \, d\mu_n(z) \to \int_K f(z) \, d\mu(z)$, then $\liminf_{n \to \infty} I(\mu_n) \geq I(\mu)$.

2. Every sequence in $P(K)$ has a weak *-convergent subsequence.

Using these facts, we choose a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $P(K)$ such that

$$I(\mu_n) \longrightarrow \inf_{\mu \in P(K)} I(\mu) \quad \text{(VII.1)}$$

as $n$ tends to $\infty$. By (2), the sequence $(\mu_n)_{n \in \mathbb{N}}$ has a subsequence, say $(\mu_{n_k})_{k \in \mathbb{N}}$ which is weak *-convergent to some $\nu \in P(K)$. Then

$$\inf_{\mu \in P(K)} I(\mu) \leq I(\nu) \leq \liminf_{n \to \infty} I(\mu_{n_k}) = \inf_{\mu \in P(K)} I(\mu)$$

which shows that $\nu$ is an equilibrium measure for $K$. \qed
Remark VII.7: If $K \subseteq \mathbb{C}$ is compact and not polar, then there is a unique equilibrium measure $\nu_K$ for $K$. Moreover, we have that $\text{supp } \nu_K$ is contained in $\partial_e K$, where $\partial_e K$ denotes the exterior boundary of $K$, i.e. the boundary of the unbounded connected component of $\mathbb{C} - K$.

**Theorem VII.8 (Frostman):** Let $K \subseteq \mathbb{C}$ be given and let $\nu$ be an equilibrium measure for $K$. Then, the following holds:

(i) For all $z \in \mathbb{C}$ we have $\Phi_\nu(z) \leq I(\nu)$,

(ii) There is some polar set $E \subseteq \partial K$ such that for all $z \in K - E$ it holds $\Phi_\nu(z) = I(\nu)$.

**Definition VII.9:** Let $E \subseteq \mathbb{C}$ be any subset. We call $\text{cap}(E) := \sup_{\mu \in P(E)} \exp(-I(\mu))$ the logarithmic capacity of $E$. Here, $P(E)$ denotes the set of all Borel probability measures $\mu$ on $\mathbb{C}$ with compact support satisfying $\text{supp } \mu \subseteq E$.

Let $K \subseteq \mathbb{C}$ be a compact set and let $\nu$ be an equilibrium measure for $K$. Note that then it holds $\text{cap}(K) = \exp(-I(\nu))$.

**Theorem VII.10:**

(i) For $E \subseteq \mathbb{C}$, we have $\text{cap}(E) = 0$ if and only if $E$ is polar.

(ii) If $E_1 \subseteq E_2 \subseteq \mathbb{C}$, then $\text{cap}(E_1) \leq \text{cap}(E_2)$.

(iii) If $E \subseteq \mathbb{C}$ and $\alpha, \beta$ are complex numbers, then $\text{cap}(\alpha E + \beta) = |\alpha| \text{cap}(E)$.

(iv) If $K \subseteq \mathbb{C}$ is compact, then $\text{cap}(K) = \text{cap}(\partial_e K)$.

(v) For a compact set $K \subseteq \mathbb{C}$, we denote by $\Omega(K)$ the connected component of $(\mathbb{C} \cup \{\infty\}) - K$ which contains $\infty$. If $K_1, K_2 \subseteq \mathbb{C}$ are compact subsets and $f: \Omega(K_1) \to \Omega(K_2)$ is a meromorphic function satisfying $f(z) = z + O(1)$ as $z \to \infty$, then $\text{cap}(K_2) \leq \text{cap}(K_1)$. If $f$ is even biholomorphic, then $\text{cap}(K_2) = \text{cap}(K_1)$.

(vi) If $K \subseteq \mathbb{C}$ is compact, then it holds

$$\text{cap}(K) \leq \frac{1}{2} \text{diam}(K), \quad \text{cap}(K) \geq \sqrt{\frac{1}{\pi} \lambda^2(K)},$$

where $\text{diam}(K) := \max\{|w_1 - w_2| \mid w_1, w_2 \in K\}$ and $\lambda$ denotes the Lebesgue measure on $\mathbb{C}$.
(vii) If $K \subseteq \mathbb{C}$ is compact and $q(z) = \sum_{k=0}^{d} a_k z^k$ with $a_d \neq 0$ is a complex polynomial, then
\[ \text{cap}(q^{-1}(K)) = \left( \frac{\text{cap}(K)}{|a_d|} \right)^{1/d}. \]

**Theorem VII.11 (Fekete-Szegö):** Let $K \subseteq \mathbb{C}$ be compact. Consider the sequence $(\delta_n(K))_{n \geq 2}$ of diameters of $K$, which was defined in exercise 2 of assignment 4B. Then $(\delta_n(K))_{n \geq 2}$ is convergent and the limit $\delta(K) := \lim_{n \to \infty} \delta_n(K)$ is given by $\delta(K) = \text{cap}(K)$.

**Proof:** From exercise 2(i) of assignment 4B, we know that $(\delta_n(K))_{n \geq 2}$ is decreasing and since $\delta_n(K) \geq 0$ for all $n \geq 2$, it follows that $(\delta_n(K))_{n \geq 2}$ is convergent. To make the proof easier to digest, we prove two smaller results first.

(1) For all $n \geq 2$, it holds that $\delta_n(K) \geq \text{cap}(K)$. To see this, let $w_1, \ldots, w_n$ in $K$ be given. By definition of $\delta_n(K)$ we have that
\[ \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \log |w_i - w_j| \leq \log \delta_n(K). \]
Hence, for every $\mu \in P(K)$, we get by integration of the latter inequality with respect to the product measure $\mu^n$ over $K^n$ that
\[ \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \int_K \cdots \int_K \log |w_i - w_j| \, d\mu(w_1) \cdots d\mu(w_n) \leq \log \delta_n(K). \]
Since for each of the $n(n-1)/2$ possible choices of indices $1 \leq i < j \leq n$ it holds
\[ \int_K \cdots \int_K \log |w_i - w_j| \, d\mu(w_1) \cdots d\mu(w_n) = \int_K \int_K \log |w_i - w_j| \, d\mu(w_i) \, d\mu(w_j) = -I(\mu), \]
we infer from the latter that $\exp(-I(\mu)) \leq \delta_n(K)$. Thus, it follows that $\text{cap}(K) = \sup_{\mu \in P(K)} \exp(-I(\mu)) \leq \delta_n(K)$, as desired.

(2) For each $n \geq 2$, let $w^{(n)} = (w_1^{(n)}, \ldots, w_n^{(n)})$ be a Fekete $n$-tuple for $K$ and define $\mu_n \in P(K)$ by $\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{w_i^{(n)}}$. Let $(\mu_n)_{n \in \mathbb{N}}$ be a subsequence of $(\mu_n)_{n \in \mathbb{N}}$ which is weak*-convergent to some $\nu \in P(K)$. Then $I(\nu) \leq -\log \delta(K)$.
For $R > 0$, we set $\log R(x) := \min\{\log(x), R\}$. Then, by monotone convergence, we find

$$I(\nu) = \lim_{R \to \infty} \int_K \int_K \log \frac{1}{|z - w|} d\nu(z) d\nu(w)$$

and thus, since $(\mu_{n_k})_{k \in \mathbb{N}}$ is weak*-convergent to $\nu$, it follows

$$I(\nu) = \lim_{R \to \infty} \lim_{k \to \infty} \int_K \int_K \log \frac{1}{|z - w|} d\mu_{n_k}(z) d\mu_{n_k}(w).$$

Next, we observe that

$$\int_K \int_K \log R \frac{1}{|z - w|} d\mu_{n_k}(z) d\mu_{n_k}(w)$$

$$= \frac{1}{n_k^2} \sum_{i,j=1}^{n_k} \log R \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|}$$

$$= \frac{2}{n_k^2} \sum_{1 \leq i < j \leq n_k} \log R \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|} + \frac{1}{n_k} R$$

$$\leq \frac{2}{n_k^2} \sum_{1 \leq i < j \leq n_k} \log \frac{1}{|w_i^{(n_k)} - w_j^{(n_k)}|} + \frac{R}{n_k} = - \frac{n_k - 1}{n_k} \log \delta_{n_k}(K) + \frac{R}{n_k}.$$

Hence, we deduce that

$$I(\nu) \leq \lim_{R \to \infty} \lim_{k \to \infty} \left(- \frac{n_k - 1}{n_k} \log \delta_{n_k}(K) + \frac{R}{n_k}\right) = - \log \delta(K),$$

as asserted.

Applying the results (1) and (2) in this order, we obtain that

$$\text{cap}(K) \leq \delta(K) \leq \exp(-I(\nu)) \leq \sup_{\mu \in P(K)} \exp(-I(\mu)) = \text{cap}(K),$$

i.e. $\delta(K) = \text{cap}(K)$, which proves the theorem.

Further, we see that $\nu$ must be an equilibrium measure for $K$. As there is a unique equilibrium measure $\nu_K$ for $K$ in the case $\text{cap}(K) > 0$, it follows from (1) that the sequence $(\mu_n)_{n \in \mathbb{N}}$ then has $\nu_K$ as its only limit point. Therefore, $(\mu_n)_{n \in \mathbb{N}}$ itself must be weak*-convergent to $\nu_K$. \qed
Chapter VIII.

Uniform Approximation

Suppose that $K \subseteq \mathbb{C}$ is a compact set for which the complement $\mathbb{C} - K$ is connected. In this situation, Runge’s Theorem says that every $f \in \mathcal{O}(\Omega)$, on some open set $\Omega \subseteq \mathbb{C}$ satisfying $K \subseteq \Omega$, can be approximated uniformly on $K$ by (holomorphic) polynomials. We prove a quantitative version of this result.

**Theorem VIII.1 (Bernstein-Walsh):** Let $K \subseteq \mathbb{C}$ be compact, suppose that $\mathbb{C} - K$ is connected and let $\nu \in P(K)$ be an equilibrium measure for $K$. Suppose that $f \in \mathcal{O}(\Omega)$, where $K \subseteq \Omega \subseteq \mathbb{C}$ is open. Put

$$\Theta := \begin{cases} \sup_{z \in (\mathbb{C} \cup \{\infty\}) - \Omega} \exp(\Phi_\nu(z) - I(\nu)), & \text{if } \text{cap}(K) > 0, \\ 0, & \text{if } \text{cap}(K) = 0, \end{cases}$$

Then $\Theta < 1$ and it holds $\limsup_{n \to \infty} d_n(f, K)^{1/n} \leq \Theta$, where

$$d_n(f, K) := \inf\{\|f - p\|_K \mid p \text{ holomorphic polynomial of degree at most } n\}.$$

The proof of this statement relies on the following result:

**Theorem VIII.2 (Bernstein’s Lemma):** In the situation of Theorem VIII.1 let $\text{cap}(K) > 0$. Then it holds:

(i) If $q$ is a polynomial with $n := \deg q \geq 1$, then for all $z \in \mathbb{C} - K$ it holds

$$\left( \frac{|q(z)|}{\|q\|_K} \right)^n \leq \exp(-\Phi_\nu(z) + I(\nu)).$$

(ii) If $q$ is a Fekete polynomial for $K$ with $n := \deg q \geq 2$, then for all $z \in \mathbb{C} - K$ it holds

$$\left( \frac{|q(z)|}{\|q\|_K} \right)^{1/n} \geq \exp(-\Phi_\nu(z) + I(\nu)) \left( \frac{\text{cap}(K)}{\delta_n(K)} \right)^{\tau(z, \infty)},$$

where $\tau := \tau_{\Omega(K)}$ is the Harnack distance on $\Omega(K) = (\mathbb{C} \cup \{\infty\}) - K$. 

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Chapter VIII. Uniform Approximation

Remark VIII.3:  (i) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^N$ be open and connected. Building on exercise 1 of assignment 3A, we define for $x, y \in \Omega$ that

$$
\tau_{\Omega}(x, y) := \inf \{ \tau > 0 \mid \forall u \in H_+(\Omega) : \tau^{-1}u(x) \leq u(y) \leq \tau u(x) \}
$$

We call $\tau_{\Omega} : \Omega \to \Omega \to [1, \infty)$ the Haranck distance on $\Omega$. One can show that $\log \tau_{\Omega} : \Omega \times \Omega \to [0, \infty)$ is a continuous semimetric on $\Omega$.

(ii) Due to (Theorem V.13), one can extend the notion of sub-, super and harmonic functions to Riemann surfaces and in particular to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For instance, the Harnack distance and the maximum principle (Theorems 3.8 and 5.7) remain true.

Proof:  (i) Without loss of generality, we may assume that $q$ is monic (i.e. $q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$). On $\mathbb{C} - K$ we declare a subharmonic function $u$ via

$$
u(z) := \frac{1}{n} \log |q(z)| - \frac{1}{n} \log \|q\|_K + \Phi_\nu(z) - I(\nu).
$$

Note that $z \mapsto 1/n \log |q(z)|$ is subharmonic on $\mathbb{C} - K$, refer to exercise 1 of assignment 1B and (Theorem V.8). Since $\Phi_\nu(z) = -\log |z| + O(1/|z|)$ (see Theorem VII.1) and $1/n \log |q(z)| = \log |z| + o(1)$ as $z \to \infty$, we see that $u$ extends to a subharmonic function on $\Omega(K)$ by $u(\infty) := -1/n \log \|q\|_K$. Now, for every $w \in \partial K$, we get by (Theorem VII.8)(i) that

$$
\limsup_{z \to w} u(z) \leq \frac{1}{n} \log |q(w)| - \frac{1}{n} \log \|q\|_K.
$$

By (Theorem V.7)(iii) (see also (Remark VIII.3)(iii)), if follows that $u \leq 0$ on $\Omega(K)$, which implies the assertion.

(ii) Note that all zeros of $q$ lie in $K$, thus, $u$ is harmonic on $\Omega(K)$ and $u \leq 0$. Hence $-u \in H_+(\Omega(K))$ and (Remark VIII.3)(i) gives that for all $z \in \Omega(K)$ it holds

$$
u(z) \geq \tau_{\Omega(K)}(z, \infty) u(\infty).
$$

By exercise 2(ii) of assignment 4B, we have the estimate

$$
u(\infty) = -I(\nu) - \frac{1}{n} \log \|q\|_K \geq -I(\nu) - \log \delta_n(K) = \log \left( \frac{\text{cap}(K)}{\delta_n(K)} \right).
$$

Putting this together, we obtain the result. \qed
Proof (of Theorem VIII.2): Suppose that \( \text{cap}(K) > 0 \). Let \( \Gamma \) be a closed contour in \( \Omega - K \) such that \( \text{Ind}_\Gamma(w) = 1 \) for all \( w \in K \) and \( \text{Ind}_\Gamma(z) = 0 \) for all \( z \in \mathbb{C} - \Omega \). By the global version of Cauchy’s Integral Formula (see Satz 7.12 in the Funktionentheorie I lecture notes), we have for all \( w \in K \) that

\[
f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w} \, dz. \tag{VIII.1}
\]

For \( n \geq 2 \), let \( q_n \) be a Fekete polynomial of degree \( n \) for \( K \) and put

\[
p_n(w) := \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{q_n(z)} \frac{q_n(w) - q_n(z)}{w - z} \, dz \tag{VIII.2}
\]

Then \( p_n \) is a polynomial with \( \deg p_n \leq n - 1 \). From Eq. (VIII.1) we deduce that for all \( w \in K \) it holds

\[
f(w) - p_n(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w} \, dq_n(z) \tag{VIII.3}
\]

hence \( d_n(f, K) \leq \| f - p_n \|_K \leq C \| q_n \|_K / \inf_{z \in \Gamma} |q_n(z)| \), where \( C \) is the constant \( C := \| f \|_{L^1(\Gamma)} / [2\pi \cdot \text{dist}(\Gamma, K)] \). By (Theorem VIII.2)(iii) we get for all \( z \in \Gamma \) that

\[
\left( \frac{\| q_n \|_K}{|q_n(z)|} \right)^{1/n} \leq \exp(\Phi_\nu(z) - I(\nu)) \left( \frac{\delta_n(K)}{\text{cap}(K)} \right)^{\tau_{\text{cap}}(z, \infty)} \leq \Theta_\Gamma \left( \frac{\delta_n(K)}{\text{cap}(K)} \right)^\alpha,
\]

where \( \Theta_\Gamma := \sup\{ \exp(\Phi_\nu(z) - I(\nu)) \mid z \in \Gamma \} \) and \( \alpha := \sup\{ \tau_{\text{cap}}(z, \infty) \mid z \in \Gamma \} \). Hence, due to (Theorem VII.11), it holds

\[
\limsup_{n \to \infty} d_n(f, K)^{1/n} \leq \limsup_{n \to \infty} C^{1/n} \Theta_\Gamma \left( \frac{\delta_n(K)}{\text{cap}(K)} \right)^\alpha = \Theta_\Gamma.
\]

Finally, we note that for all \( \varepsilon > 0 \), there is a contour \( \Gamma \) as above such that \( 0 \leq \Theta_\Gamma - \Theta < \varepsilon \). This proves the fact \( \limsup_{n \to \infty} d_n(f, K)^{1/n} \leq \Theta \), if \( K \) has positive capacity.

Note that \( \Theta < 1 \), because otherwise, there was a \( z_0 \in \mathbb{C} - K \) such that \( \Phi_\nu(z_0) = I(\nu) \) and thus, by Theorem VII.7(i), \( z_0 \) was a local maximum so that \( \Phi_\nu \equiv I(\nu) \) on \( \mathbb{C} - K \) by Theorem V.7(ii). But this contradicted Theorem VII.1(i), as \( I(\nu) < \infty \).

The assertion in the case \( \text{cap}(K) = 0 \) follows by approximation of \( K \) with a decreasing sequence \( (K_k)_{k \in \mathbb{N}} \) of non-polar compact subsets of \( \Omega \) satisfying \( K = \bigcap_{k=1}^\infty K_k \) from the already proved case. \( \square \)
Remark VIII.4: We notice that the polynomials $p_n$ defined in Eq. (VIII.2) satisfy for $1 \leq j \leq n$ that

$$p_n(w_j) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{q_n(z)} \frac{q_n(w_j) - q_n(z)}{w_j - z} \, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - w_j} \, dz = f(w_j),$$

where, in the second step, we have used that $q_n(w_j) = 0$, and, in the last step, Cauchy’s integral formula as formulated in Eq. (VIII.1). In other words, $p_n$ solves the following interpolation problem:

Find a holomorphic complex polynomial $p$ with $\deg p \leq n - 1$ such that for $1 \leq j \leq n$ it holds $p(w_j) = f(w_j)$.

Note that if $w_1, \ldots, w_n$ are all distinct, then $p_n$ is the unique solution of this interpolation problem. In this case, one can use the so-called Lagrange polynomials to find an explicit expression for $p_n$.

Among all holomorphic complex polynomials $p$ satisfying $\deg p \leq n$, there is always at least one best approximation $p_*$ to $f$, i.e. $p_*$ satisfies the condition $d_n(f, K) = \|f - p_*\|_K$. In general, the polynomials $p_n$ defined in Eq. (VIII.2) do not provide best approximations to $f$. Therefore, it seems plausible that a better choice of $p_n$ might lead to better results about the asymptotic behaviour of $d_n(f, K)$ as $n \to \infty$. However, one can show that for $n \geq 2$ it always holds

$$\|f - p_n\|_K \leq (n + 1)d_n(f, K).$$

Thus, we see that $\limsup_{n \to \infty} d_n(f, K)^{1/n} \leq \Theta$, namely the conclusion of (Theorem VIII.1), holds if and only if $\limsup_{n \to \infty} \|f - p_n\|_K^{1/n} \leq \Theta$.

Example VIII.5: Fix $z_0 \in \mathbb{C}$ and $r_0 > 0$ and put $K := \text{cl}(D(z_0, r_0))$. The capacity of $K$ can be shown to be $\text{cap}(K) = r_0$ and the (unique) equilibrium measure $\nu_K$ is given by $\nu_K = (2\pi r_0)^{-1} \sigma_{\partial D(z_0, r_0)}$. One finds that the associated logarithmic potential $\Phi_{\nu_K}$ is of the form

$$\Phi_{\nu_K}(z) = \begin{cases} \log \frac{1}{r_0}, & \text{if } |z - z_0| \leq r_0, \\ \log \frac{1}{|z - z_0|}, & \text{if } |z - z_0| > r_0. \end{cases}$$

Now, for any $r > r_0 > 0$, we consider $\Omega := D(z_0, r)$. Then,

$$\Theta = \sup \{ \exp (\Phi_{\nu_K}(z) - I(\nu)) \mid z \in (\mathbb{C} \cup \{\infty\}) - \Omega \} = \frac{r_0}{r}.$$
Hence, Theorem VIII.1 asserts that \( \limsup_{n \to \infty} d_n(f, K)^{1/n} \leq \Theta \) for every \( f \in \mathcal{O}(\Omega) \). This is in keeping with the rate of approximation of \( f \) by its Taylor polynomials at the point \( z_0 \). In fact, if we put

\[
T_n(w) := \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(w - z_0)^k
\]

for every integer \( n \geq 0 \), then Cauchy’s Integral Formula yields for every \( r_0 < \rho < r \) and all \( w \in K \) that

\[
f(w) - T_n(w) = \frac{1}{2\pi i} \int_{\partial D(z_0, \rho)} f(\zeta) \left( \frac{1}{\zeta - w} - \sum_{k=0}^{n} \frac{(w - z_0)^k}{(\zeta - z_0)^{k+1}} \right) d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\partial D(z_0, \rho)} f(\zeta) \sum_{k=n+1}^{\infty} \frac{(w - z_0)^k}{(\zeta - z_0)^{k+1}} d\zeta.
\]

We infer from the latter that \( \|f - T_n\|_{K} \leq \|f\|_{\partial D(z_0, \rho)} (1 - \frac{r_0}{\rho})^{-1}(\frac{r}{\rho})^{n+1} \), which yields

\[
\limsup_{n \to \infty} \|f - T_n\|_{K}^{1/n} \leq \frac{r_0}{\rho}.
\]

As \( r_0 < \rho < r \) was arbitrary, we can let \( \rho \uparrow r \) and obtain the estimate \( \limsup_{n \to \infty} \|f - T_n\|_{K}^{1/n} \leq \Theta \).

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**Figure VIII.1.** Graph of the potential \( \Phi_{\nu_K} \) for the equilibrium measure \( \nu_K \) for \( K = \text{cl}(D(z_0, r_0)) \) with \( z_0 = 1 \) and \( r_0 = 2 \), see Example VIII.5.
Chapter VIII. Uniform Approximation

Example VIII.6: For the interval $K = [-1, 1]$, one can show that $\text{cap}(K) = 1/2$ and that the (unique) equilibrium measure is given by $d\nu_K(x) = (1 - x^2)^{-2} \, dx$. Furthermore, one obtains that

$$\Phi_{\nu_K}(z) = \begin{cases} \log(2), & \text{if } z \in [-1, 1], \\ \log(2) - \log |z + (z^2 - 1)^{1/2}|, & \text{if } z \in \mathbb{C} - [-1, 1]. \end{cases}$$

Figure VIII.2.: Graph of the potential $\Phi_{\nu_K}$ for the equilibrium measure $\nu_K$ for $K = [-1, 1]$, see Example VIII.6.