Seminar: Millenium Problems
The Hodge conjecture 29.11.2017

The conjecture:
On a projective non-singular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes $cl(Z)$ of algebraic cycles.

1.1 Definition: Let $(A_n)_{n\in\mathbb{Z}}$ $[(A^n)_{n\in\mathbb{Z}}]$ be a sequence of $\mathbf{R}$-modules and $(d_n)_{n\in\mathbb{Z}}$ $[(d^n)_{n\in\mathbb{Z}}]$ a sequence of homomorphisms $d_n : A_n \to A_{n-1}$ $[d^n : A^n \to A^{n+1}]$ such that $d_n \circ d_{n+1} = 0$ $[d^n \circ d^{n-1} = 0]$.
Then (A_*, d_*) $[(A^*, d^*)]$ is called a *[co]chain complex* with *boundary operator* d_n $[d^n]$.
The elements of A_n $[A^n]$ are called *n-[co]chains*, the elements of $Z_n := \ker d_n \subseteq A_n$ $[Z^n := \ker d^n \subseteq A^n]$ *n-[co]cycles* and the elements of $B_n := \operatorname{im} d_{n+1} \subseteq A_n$ $[B^n := \operatorname{im} d^{n-1} \subseteq A^n]$ *n-[co]boundaries.*

1.2 Definition: Let (A, d) be a (co)chain complex. The quotient $H_n(A, d) := Z_n(A, d)/B_n(A, d)$ is called the *n-th (co)homology class* of (A, d), its elements *(co)homology classes* and the sequence $(H_n(A, d))_{n\in\mathbb{Z}}$ the *(co)homology* of (A, d). Two elements from $H_n(A, d)$ are called *(co)homologuous.*

2.1 Definition: Let E be a vector space on $\mathbb{R}$. A *multilinear form of degree p* is a map $\alpha : E\times\cdots\times E \longrightarrow \mathbb{R}$ that is linear in every component.
α is *alternating* if $\alpha(\dots, v_i, \dots, v_j, \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)\ \forall i \neq j$.

2.2 Definition: Let $U \subseteq \mathbb{R}^n$. A *differential form of degree p* on U is a map

$$\omega : U \longrightarrow \{\text{alternating multilinear forms of degree } p \text{ on } \mathbb{R}^n\}$$

The *exterior derivative* $d\omega$ of a differential form $\omega = \sum_{1\leq i_1<\ldots<i_p\leq n} g_{i_1,\ldots,i_p} dx_{i_1}\cdots dx_{i_p}$ is defined as

$$d\omega := \sum_{1\leq i_1<\ldots<i_p\leq n} (\frac{\partial}{\partial x_1} g dx_1 + \cdots + \frac{\partial}{\partial x_n} g dx_n) dx_{i_1}\cdots dx_{i_p}$$

A $(p-1)$-differential form α on U is called a *primitive function* of ω, if $d\alpha = \omega$.
ω is called *exact*, if it has a primitive function and *closed*, if $d\omega = 0$.

3.1 Definition: A function $f : X \to Y$, X, Y topological spaces, is called a *homeomorphism*, if f is bijective, continuous and f^{-1} is continuous. A topogical space M together with a family $(U_i, \phi_i)_{i\in I}$ of open subsets U_i and homeomorphisms $\phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{R}^n$, such that $M = \bigcup_{i\in I} U_i$, is called a *topological manifold.* ϕ is called a *chart*, $(U_i, \phi_i)_{i\in I}$ an *atlas.* The *transition maps* of an atlas are

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}\big|_{\phi_\alpha(U_\alpha\cap U_\beta)} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

4.1 Definition: Let X be a smooth manifold and $\Omega^p(X)$ the set of smooth p-differential forms on X. The *de-Rham-complex* is the cochain complex

$$0 \to C^\infty(X) \cong \Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \xrightarrow{d^2} \cdots$$

where $d^p : \Omega^p(X) \to \Omega^{p+1}(X)$ is the exterior derivative.
The *k-th de-Rham-cohomology* $H^k_{dR}(X)$ is the k-th cohomology group of the de-Rham-complex.

4.2 Theorem: (de Rham, 1931)
The de-Rham-cohomology $H^*_{dR}(X)$ of a smooth manifold X is isomorph to the singular cohomology in R, i.e. $H^*_{dR} \cong H^*_{sing}(X, \mathbb{R})$.
For $c \in H^{sing}_p(X)$ the isomorphism is given by

$$\omega \in H^p_{dR}(X) \longmapsto (c \mapsto \int_c \omega) \in (H^{sing}_p(X, \mathbb{R}) \cong H^*_{sing}(X, \mathbb{R})$$