Seminar: Millenium Problems
The Hodge conjecture 06.12.2017

The conjecture:
On a projective non-singular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes $cl(Z)$ of algebraic cycles.

1 Definition. A *complex manifold* is a topological space M with an atlas $(U_i)_{i\in I}$ of open sets in M covering M and a set of coordinate charts $\Phi_i : U_i \to D_i \subseteq \mathbb{C}^n$ such that the transition maps $g_{i,j} = \Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$ are *biholomorphic.*

2 Definition. Let M be a complex manifold. A *differential form of type* (p, q) is a differential form that can be written as

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \varphi_{I,J} dz_I d\bar{z}_J$$

The space of the (p, q)-forms on M is denoted by $A^{p,q}(M)$

3 Remark. The exterior derivative operator d can be decomposed as $d = \partial + \bar{\partial}$, where
$\partial : A^{p,q}(M) \to A^{p+1,q}(M)$ and $\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$ are the so-called *Dolbeault operators.*
The *Dolbeault complex* is the complex

$$0 \to A^{p,0}(M) \xrightarrow{\bar{\partial}} A^{p,1}(M) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} A^{p,n}(M) \to 0$$

and the q-th cohomology class from this complex is called the *Dolbeault cohomology class*:

$$H^{p,q}(M) = \frac{ker\,\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)}{im\,\bar{\partial} : A^{p,q-1}(M) \to A^{p,q}(M)}$$

4 Proposition. *Let M be a compact Kähler manifold. Then, the* Hodge decomposition *holds, i.e. there exists an isomorphism*

$$\bigoplus_{p+q=n} H^{p,q}(M) \longrightarrow H^n(M, \mathbb{C}) = H^n(M, \mathbb{R}) \otimes \mathbb{C}$$

where $H^n(M, \mathbb{R})$ is the n-th de Rham cohomology class of M.

5 Definition. A *Hodge cycle* is an elelment of the cohomology class $H^{2p}(M, \mathbb{Q})$ which under the Hodge decomposition corresponds to an elelment of the Dolbeault cohomology class $H^{p,p}(M)$
A *Hodge class* is the cohomology class of such Hodge cycles: $Hdg^p(M) = H^{2p}(M, \mathbb{Q}) \cap H^{p,p}(M)$

6 Definition. A *(closed) projective set* $X \subseteq \mathbb{P}^n$ is a subset of the projective space, that can be written as $X = V(S) = \{x \in \mathbb{P}^n | f(x) = 0 \forall f \in S\}$ for S a subset of the homogeneous polynomial functions $\mathbb{C}[X_0, X_1, \ldots, X_n]$

7 Definition. For a projective set X an open subset is a set of the form $X\backslash Y$ where Y is a projective set in the same projective space. These open subsets define the *Zariski topology* on X.

8 Definition. A *projective variety* is a projective set together with the Zariski topology and the ring of regular functions.

9 Definition. An irreducible variety X is *nonsingular* at $x \in X$ if the dimension of the (embedded) tangent space at x equals the dimension of X: $dim\, T_x X = dim X$, else *singular.*
X is called *nonsingular* if it is nonsingular at all points.

10 Definition. An *algebraic cycle* of an algebraic variety X is a formal linear combination $V = \sum_i c_i Z_i$ of irreducible, closed subvarieties Z_i.