The Navier Stokes Equations

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1 Partial Differential Equations

Definition 1.1. Let U denote an open subset of $\mathbb{R}^n$. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0$$

for all $x \in U$ is called a k^{th} **order partial differential equation (PDE)** where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is given and $u : U \to \mathbb{R}$ is unknown.

Definition 1.2. An expression of the form

$$F(D^k \mathbf{u}(x), D^{k-1}\mathbf{u}(x), \ldots, D\mathbf{u}(x), \mathbf{u}(x), x) = 0$$

for all $x \in U$ is called a k^{th} **order system of PDEs** where $F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$ is given and $\mathbf{u} : U \to \mathbb{R}^m$ with $\mathbf{u} = (u_1, \ldots, u_m)$ is unknown.

Definition 1.3. A given problem is **well-posed** if

(a) the problem in fact has a solution,

(b) this solution is unique,

(c) the solution depends continuously on the data given in the problem.

Otherwise, it is called **ill-posed**.

A **classical** or **strong solution** is a u verifying the PDE(s) (of order k) which is k times continuously differentiable such that conditions (a), (b) and (c) hold.

A **generalized** or **weak solution** satisfies the equation(s) in some precisely defined sense. It can occur that a PDE does not have a (continuously) differentiable solution, but we may find a weaker form of solution by reformulating the problem in a weaker way.

2 Navier Stokes Equations

- a semilinear system of PDEs of second order
- describing the motion of a fluid in $\mathbb{R}^n$ $(n = 2, 3)$

$$\left.\begin{array}{c} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i \\[2ex] \mathrm{div} u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \end{array}\right\} (NSE)$$

Given

- $x \in \mathbb{R}^n$ and $t \geqslant 0$
- ν viscosity of the fluid
- $f_i(x,t)$ components of given, externally applied forces

Looking for

- $p(x,t) \in \mathbb{R}$ pressure
- $u(x,t) = (u_i(x,t))_{i=1,\ldots,n} \in \mathbb{R}^n$ velocity vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)^{\mathsf{T}} \text{ (Nabla operator)}$$

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \text{ (Laplacian)}$$

Do smooth, physically reasonable solutions for the Navier Stokes equations in $\mathbb{R}^3$ exist? Here, physically reasonable means that the solutions do not blow up for $t \in [0, T]$. So we need $p, u \in C^\infty(\mathbb{R}^n \times [0, \infty))$ and $\int_{\mathbb{R}^n} |u(x,t)|^2 \mathrm{dx} < C$ for all $t \geqslant 0$ (bounded energy).