The Navier Stokes Equations

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1 Navier Stokes Equations

Definition 1.1. The **Navier-Stokes-Equations(NSE)** describe the motion of a fluid in $\mathbb{R}^n$ and are given by:

$$\frac{\partial u_i}{\partial t} + (u_i \cdot \bigtriangledown) u_i = v \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t)$$
$$div\ u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0$$

with initial conditions $u(x,0) = u_0(x)$ and a Dirichlet BC $u(x,t) = \Phi(x,t)$ on $\delta\Omega \times [0,\infty)$.

Definition 1.2. Formulation of the Clay Institute

1.) **Existence and smoothness of NSE solutions on $\mathbb{R}^3$:**
Take $v > 0$ and $n = 3$. Let $u_0(x)$ be any smooth, divergence-free vector field. Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy the NSE and are physically reasonable.

2.) **The Breakdown of** 1.)

Question: What is a weak solution of the NSE?
Idea: Integrate the first equation against a test function $\Phi(x,t) = (\Phi_i(x,t))_{1 \leqslant i \leqslant n}$ and the second equation also against a test function $\phi(x,t)$ $(\forall \Phi, \phi)$. Then use integration by parts to make the derivatives fall on the test function. We receive the weak formulation of our PDE and their solutions are called **weak solutions**:

2 Problems in $\mathbb{R}^3$

The formulation 1.) of the Clay Institute has been proven by O. Ladyzhenskaya for $n = 2$ due to:

- a simplification of the conservation of energy
- the fact that most problems of the three-dimensional case are just absent for $n = 2$

Main difficulties for $n = 3$:

- to pass the point of blow-up time T: velocity $(u_i(x,t))_{1 \leqslant i \leqslant 3}$ becomes unbounded at $T \Rightarrow$ no chance to solve the problem in the form of a physically reasonable solution
- turbulence: describes the rotation and motion of a particle due to collision with other particles.

$$\int_0^T \sup_{x \in \mathbb{R}^3} |curl\ u(x,t)| dt = \infty$$

$\Rightarrow$ no method known to solve such a nonlinear PDE without any additional conditions

3 strategies remaining:
S1.) Solve the NSE exactly and explicitly
S2.) Discover a new globally controlled quantity ($\rightarrow$keywords: coercive and critical)
S3.) Show existence of global smooth solutions (even without any additional conditions)

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} u \frac{\partial \Phi}{\partial t} \,\mathrm{d}x\,\mathrm{d}t - \sum_{ij} \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_i u_j \frac{\partial \Phi_i}{\partial x_j} \,\mathrm{d}x\,\mathrm{d}t = v \int_{\mathbb{R}^3} \int_{\mathbb{R}} u \Delta \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3} \int_{\mathbb{R}} f \phi \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^3} \int_{\mathbb{R}} p\, (div \Phi) \,\mathrm{d}x\,\mathrm{d}t$$
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} u \bigtriangledown_x \phi \,\mathrm{d}x\,\mathrm{d}t = 0$$

3 Special cases of the NSE