

The Navier Stokes Equations

Philipp Schmeer , 20.12.2017

1 Navier Stokes Equations

Definition 1.1. The **Navier-Stokes-Equations(NSE)** describe the motion of a fluid in \mathbb{R}^n and are given by:

$$\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x, t)$$

$$\operatorname{div} u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0$$

with initial conditions $u(x, 0) = u_0(x)$ and a Dirichlet BC $u(x, t) = \Phi(x, t)$ on $\delta\Omega \times [0, \infty)$.

Definition 1.2. Formulation of the Clay Institute

1.) Existence and smoothness of NSE solutions on \mathbb{R}^3 :

Take $\nu > 0$ and $n = 3$. Let $u_0(x)$ be any smooth, divergence-free vector field. Take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t), u_i(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the NSE and are physically reasonable.

2.) The Breakdown of 1.)

Question: What is a weak solution of the NSE?

Idea: Integrate the first equation against a test function $\Phi(x, t) = (\Phi_i(x, t))_{1 \leq i \leq n}$ and the second equation also against a test function $\phi(x, t)$ ($\forall \Phi, \phi$). Then use integration by parts to make the derivatives fall on the test function. We receive the weak formulation of our PDE and their solutions are called **weak solutions**:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} u \frac{\partial \Phi}{\partial t} dx dt - \sum_{ij} \int_{\mathbb{R}^3} \int_{\mathbb{R}} u_i u_j \frac{\partial \Phi_i}{\partial x_j} dx dt = \nu \int_{\mathbb{R}^3} \int_{\mathbb{R}} u \Delta \Phi dx dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}} f \phi dx dt - \int_{\mathbb{R}^3} \int_{\mathbb{R}} p (\operatorname{div} \Phi) dx dt$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} u \nabla_x \phi dx dt = 0$$

2 Problems in \mathbb{R}^3

The formulation 1.) of the Clay Institute has been proven by O. Ladyzhenskaya for $n = 2$ due to:

- a simplification of the conservation of energy
- the fact that most problems of the three-dimensional case are just absent for $n = 2$

Main difficulties for $n = 3$:

- to pass the point of blow-up time T : velocity $(u_i(x, t))_{1 \leq i \leq 3}$ becomes unbounded at $T \Rightarrow$ no chance to solve the problem in the form of a physically reasonable solution
- turbulence: describes the rotation and motion of a particle due to collision with other particles.

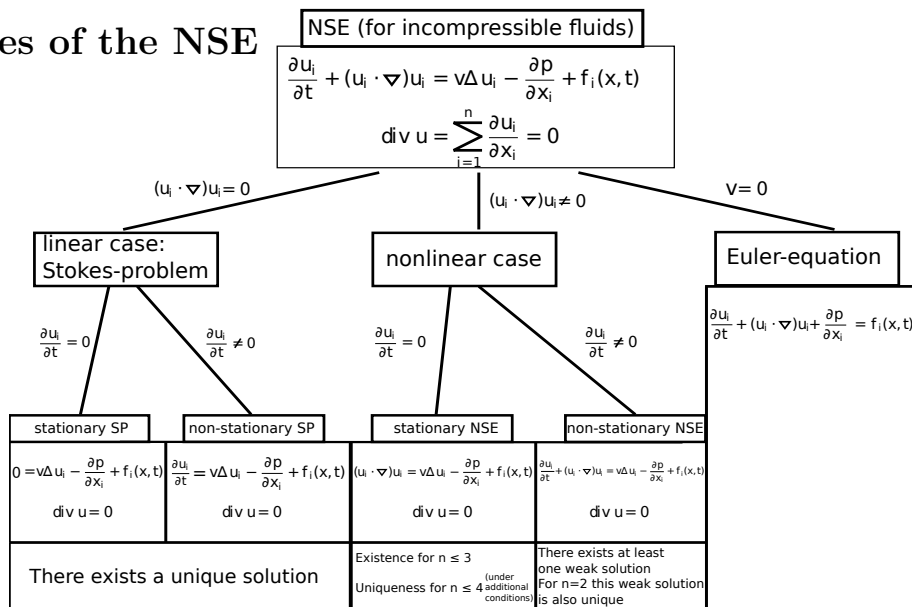
$$\int_0^T \sup_{x \in \mathbb{R}^3} |\operatorname{curl} u(x, t)| dt = \infty$$

\Rightarrow no method known to solve such a nonlinear PDE without any additional conditions

3 strategies remaining:

- S1.)** Solve the NSE exactly and explicitly
- S2.)** Discover a new globally controlled quantity (\rightarrow keywords: coercive and critical)
- S3.)** Show existence of global smooth solutions (even without any additional conditions)

3 Special cases of the NSE



$$\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i + \frac{\partial p}{\partial x_i} = f_i(x, t)$$