Summer term 2019
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Set theory & Forcing

Introduction

Motivation

- 19th Century: dealing with infinitely small objects in analysis
  - ⇝ what *are* the real numbers?
  - ⇝ what is a set? What is infinity?
- Cantor 1895: "A set is any collection into a whole of definite and separate objects of our intuition or our thought."
- Russell 1903: $R := \{x \mid x \notin x\}$

  Then $R \in R \iff R \notin R$

  ⇝ crisis of mathematics (1903 – 1931):

  What are our axioms? What is a set??

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können. D. Hilbert

- Hilbert 1900, 2nd problem: Are the Peano axioms free from contradictions? (i.e. "consistent")

  5 axioms to define $\mathbb{N}$
- Gödel 1931: No system can prove its own consistency. (2nd Incompleteness Thm) (So, no, David!)

Set theory

- Zermelo – Fraenkel 1907/1921: V is the universe of all sets (involved in Russell 1903)

  satisfying the axioms ZFC (= ZF + (axiom of choice (AC))),

  i.e. $x \in V \iff$ "x is a set"
- class: $A := \{x \mid \varphi(x)\}$, φ statement

  $x \in V \not\Leftarrow x$ class true class: V, R, Card, On

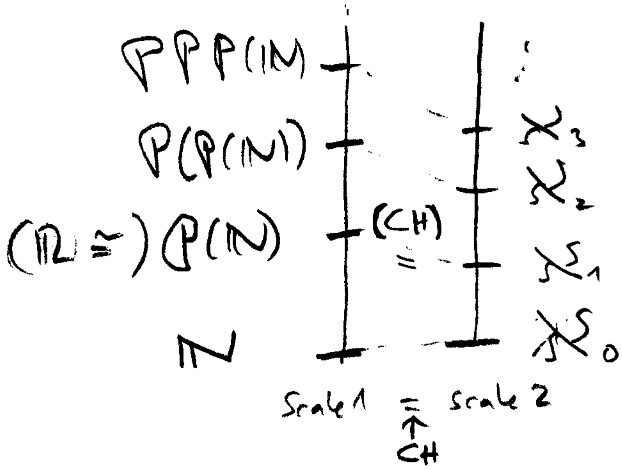
  Card := all cardinal numbers $\exists f : u \longleftrightarrow v$ bijective

  (wrt $\in$) On := all ordinal numbers (cardinalities of well-orderings)

Continuum hypothesis

proved
$\mathbb{N} \cong \mathbb{Q}$
$\mathbb{N} \not\cong \mathbb{R}$
ZF
Fraenkel

- Cantor 1878: There is no set whose cardinality is strictly between $\mathbb{N}$ and $\mathbb{R}$. (CH)
- In other words, a question of the scale of infinity:



- Gödel 1938: $\exists$ constructive universe L and

$$ZF + V = L \vdash AC$$
$$ZF + V = L \vdash (G)CH \text{, even } \mathcal{P}(\aleph_\alpha) = \aleph_{\alpha+1}$$
$$\mathrm{Cons}(ZF) \Rightarrow \mathrm{cons}(ZF + V = L)$$
($\nexists$ contradiction) $\Rightarrow \mathrm{cons}(ZFC)$, $\mathrm{cons}(ZF + CH)$

Forcing (Cohen 1963, Feldsmedal 1966)

- Construct a model $M[G] := \{$interpretations of elements from $M^{\mathbb{P}}\}$, where $\mathbb{P}$ is a partial ordering, M is a model, G is a filter such that $M[G] \models$ certain statements,
  i.e. $M[G]$ is a model of ZF(C) of which we have good control:
  $M[G]$ is a different (but ZFC consistent) interpretation of "What is a set?"
- $\mathrm{Cons}\ ZFC \Rightarrow \mathrm{cons}\ ZFC + \neg CH$ (even $\mathrm{Cons}\ ZFC + 2^\omega \geq \aleph_\alpha$ $\forall \alpha \in On$
  arbitrary blow up)
  $\mathrm{Cons}\ ZF \Rightarrow \mathrm{Cons}\ ZF + \neg AC$
- Hence:
  - CH is independent from ZFC
  - AC is independent from ZF

Applications of forcing

- Baire theory: $K \subseteq \mathbb{R}$ closed nowhere dense (cnwd)
  $:\Longleftrightarrow K$ closed, empty interior
  (eg K finite, K Cantor set)
  Thm (Baire): $\mathbb{R} \neq \bigcup_{i \in \aleph_0} K_i$, K_i cnwd
  Q: $\mathbb{R} = \bigcup_{i \in \aleph_1} K_i$, K_i cnwd ? indep. from ZFC !
  ($\underset{\Downarrow}{\text{ZFC} + \text{CH}} \vdash$ yes)
  $\aleph_1 = \mathbb{R}$

- Lebesgue theory: $\mathbb{R} \neq \bigcup_{i \in \aleph_0} K_i$, K_i null set (since $\bigcup_{i \in \aleph_0} K_i$ null set)
  Q: $\mathbb{R} = \bigcup_{i \in \aleph_1} K_i$, K_i null set ? indep. from ZFC!
  (ZFC + CH $\vdash$ yes)

- $\mathcal{B}(H) := \{T : H \to H$ bounded, linear$\}$, H sep. Hilbert space
  $\mathcal{K}(H) := \{$compact operators$\} \lhd \mathcal{B}(H)$ closed two-sided ideal
  $\mathcal{Q}(H) := \mathcal{B}(H) / \mathcal{K}(H)$ Calkin algebra
  $\varphi : \mathcal{Q}(H) \to \mathcal{Q}(H)$ Automorphism $:\Longleftrightarrow \varphi$ bij., involutive algebra hom.
  φ inner $:\Longleftrightarrow \exists u \in \mathcal{Q}(H)$ unitary $(u^*u = uu^* = 1) : \varphi(x) = uxu^*$
  φ outer $:\Longleftrightarrow \varphi$ $\neg$ inner
  Phillips-Weaver 2007 (arXiv 2006): The Calkin algebra has outer automorphisms
  Farah 2011 (arXiv 2007): All automorphisms of the Calkin algebra are inner
  Contradiction? No: PW: ZFC + CH $\vdash$ $\mathcal{Q}(H)$ has outer autom.
  F: ZFC + TA $\vdash \neg($ — $)$

Literature

- Kunen, Set Theory, 2011 – good intuition / background / introduction (like Baire/Lebesgue)
- Jech, Set Theory, 1978 – advanced on forcing, no basics
- (Hausdorff, Grundzüge der Mengenlehre, 1914 – also Hausdorff space, Banach-Tarski paradox)

Schedule / Contents

I Introduction to logical calculus & Gödel's incompleteness theorems

16 April } primitive recursive, Gödel numbers/coding, ...
23 April }
30 April } formal languages, PL1, logical calculus
7 May }
14 May } Model theory
21 May } Gödel's incompleteness theorems Gödel 1931

II Set theory and forcing

28 May } ordinal/cardinal numbers, ZF
4 June }
11 June } (AC), ZF + $V = L \vdash AC, CH$ Gödel 1938
18 June } Forcing
25 June }
2 July } no lecture
9 July } cons ZFC $\Rightarrow$ cons ZFC + $\neg$CH Cohen 1963
16 July } Applications like for the Calkin algebra

Exercise sessions / credit points

"every second week", talk in one of these sessions