# I Introduction to logical calculus & Gödel's incompleteness theorems

## §1 Primitive recursive functions and predicates

1.1 Def.: The class PRT of <u>primitive recursive terms</u> is defined by:
(i) The symbol $\underline{S}$ is a 1-ary PRT. ($\underline{S} \in PRT^1$)
(ii) For all $n, k \in \mathbb{N}$, $n \geq 1$, the symbol $\underline{C}^n_k$ is an $n$-ary PRT. ($\underline{C}^n_k \in PRT^n$)
(iii) For $n \in \mathbb{N}$, $n \geq 1$, $1 \leq k \leq n$, the symbol $\underline{P}^n_k$ is an $n$-ary PRT.
(iv) Given $\underline{g}_1, \ldots \underline{g}_m$ $n$-ary PRTs and $\underline{h}$ an $m$-ary PRT, then $\underline{Sub}(\underline{h}, \underline{g}_1, \ldots \underline{g}_m)$ is an $n$-ary PRT.
(v) Given $\underline{g}$ $n$-ary PRT, $\underline{h}$ $(n+2)$-ary PRT, then $\underline{Rec}(\underline{g}, \underline{h})$ is an $(n+1)$-ary PRT.

1.2 Def.: Given $\underline{f} \in PRT^n$, we define $val(\underline{f}) : \mathbb{N}^n \to \mathbb{N}$ by:
(i) $val(\underline{S}) : \mathbb{N} \to \mathbb{N}$, $x \mapsto x+1$ Successor
(ii) $val(\underline{C}^n_k) : \mathbb{N}^n \to \mathbb{N}$, $(x_1, \ldots x_n) \mapsto k$ Constant
(iii) $val(\underline{P}^n_k) : \mathbb{N}^n \to \mathbb{N}$, $(x_1, \ldots x_n) \mapsto x_k$ projection
(iv) $g_i = val(\underline{g}_i) : \mathbb{N}^n \to \mathbb{N}$, $h = val(\underline{h}) : \mathbb{N}^m \to \mathbb{N}$,
then $val(\underline{Sub}(\underline{h}, \underline{g}_1, \ldots \underline{g}_m)) : \mathbb{N}^n \to \mathbb{N}$,
$(x_1, \ldots x_n) \mapsto h(g_1(x_1 \ldots x_n), \ldots g_m(x_1 \ldots x_n))$, Composition
(v) $g = val(\underline{g}) : \mathbb{N}^n \to \mathbb{N}$, $h = val(\underline{h}) : \mathbb{N}^{n+2} \to \mathbb{N}$, then
$val(\underline{Rec}(\underline{g}, \underline{h})) : \mathbb{N}^{n+1} \to \mathbb{N}$, $(x_0, x_1, \ldots x_n) \overset{f}{\mapsto} \begin{cases} g(x_1, \ldots x_n) & x_0 = 0 \\ h(k, f(k, x_1, \ldots x_n), x_1, \ldots x_n) & x_0 = k+1 \end{cases}$
Recursion

1.3 Remark: (a) The recursion function is well-defined (and Sub, too).
(b) There are terms $\underline{f} \neq \underline{g}$ with $val(\underline{f}) = val(\underline{g})$. (infinitely many; Thm by Rice)
intensionally defined (by algorithm) — extensionally defined (by their values)

**1.4 Def.:** Inductive definition of digits:

(a) $\underline{0}$ is a digit

(b) If $\underline{n}$ is a digit, then also $(\underline{S}\,\underline{n})$.

Interpretation: $val(\underline{0}) := 0$, $val(\underline{S}\,\underline{n}) := val(\underline{S})(val(\underline{n})) = n+1$.

**1.5 Remark:** Can now define $\underline{f}\,\underline{z_1} \dots \underline{z_n}$ for $\underline{f} \in PRT^n$, $\underline{z_1}, \dots, \underline{z_n}$ digits.

**1.6 Def.:** A function $f : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive (PRF), if there is $\underline{f} \in PRT$ with $f = val(\underline{f})$.

**1.7 Examples:** (a) $\underline{\oplus} = \underline{Rec}(\underline{P}^1_1, \underline{Sub}(\underline{S}, \underline{P}^3_2))$ yields $val(\oplus)(x_0, x_1) = x_0 + x_1$

Addition PRF

$$val(\oplus)(0, x_1) = P^1_1(x_1) = x_1 = 0 + x_1$$
$$val(\oplus)(k+1, x_1) = S(+(k, x_1)) = (k+1) + x_1$$

(b)
$$val(\odot)(0, x_1) = 0 = 0 \cdot x_1$$
$$val(\odot)(k+1, x_1) = k x_1 + x_1 = (k+1) \cdot x_1$$

Multiplication PRF

$$\underline{\odot} = \underline{Rec}(\underline{C}^1_0, \underline{Sub}(\underline{\oplus}, \underline{P}^3_2, \underline{P}^3_3))$$

(c) $\exp$, $n!$, $\dot{-}$, $|a-b|$, ... PRF

**1.8 Def.:** (a) An n-ary predicate is a subset $P \subseteq \mathbb{N}^n$.
We write $P(z_1, \dots, z_n) :\Longleftrightarrow (z_1, \dots, z_n) \in P$.

(b) A predicate is primitive recursive (PRP), if its characteristic function is PRF.

**1.9 Examples:** $a = b$, $a < b$, ... are primitive recursive

Likewise $a \neq b$, $a \leq b$ etc since $P, Q \in PRP \Rightarrow P^c (= \neg P), P \cap Q, P \cup Q \in PRP$.

**1.10 Theorem:** The class PRP is closed under:

(a) Boolean operations $\neg, \wedge, \vee$ (i.e. $\neg P := P^c$, $P \cap Q$, $P \cup Q$)

(b) $\exists^b P = \{(k, x_1, \dots, x_n) \mid \exists y \leq k : P(y, x_1, \dots, x_n)\}$
$\forall^b P = \{(k, x_1, \dots, x_n) \mid \forall y \leq k : P(y, x_1, \dots, x_n)\}$

generalized Boolean operations

(c) Substitution with prim. rec. functions.

**Proof:** (b) $(k, x_1, \dots, x_n) \in \exists^b P \Longleftrightarrow P(0, x_1, \dots, x_n) \vee \dots \vee P(k, x_1, \dots, x_n)$
$\Longleftrightarrow \sum_{i=0}^{k} \chi_P(i, x_1, \dots, x_n) \neq 0$ $\square$

**1.11 Def:** Let $p : \mathbb{N} \to \mathbb{N}$ be the numbering of all prime numbers.

(a) $\langle z_1, \dots z_k \rangle := p(0)^{z_1+1} \cdot \dots \cdot p(k)^{z_k+1}$ is the code of $(z_1, \dots z_k) \in \mathbb{N}^k$

(b) $seq := \{ x \in \mathbb{N} \mid \exists z_1, \dots z_k \in \mathbb{N} : x = \langle z_1, \dots z_k \rangle \} \cup \{0\}$ $\quad length(\langle z_1, \dots z_k \rangle) := k$

(c) For $x \in seq$, $x = \langle z_1, \dots z_k \rangle$ is $(x)_i := z_i$ the decoding

**1.12 Lemma:** (a) The predicate $x | y$ ($x$ divides $y$) is a PRP.

(b) $Prim(x) :\Longleftrightarrow x$ is a prime number is a PRP.

(c) The function $p$ is PRF, $seq \in PRP$.

(d) Coding and decoding is PRF, $length \in PRF$

**Proof:** (a) $x | y \Longleftrightarrow \exists z \leq y : y = x \cdot z$

(b) $Prim(x) \Longleftrightarrow x \neq 0 \wedge x \neq 1 \wedge \forall y \leq x \, (\neg(y|x) \vee y = 1 \vee y = x)$

(c) Use $p(x+1) \leq p(x)! + 1$ and Thm 1.10 (b).

(d) $seq \in PRP$ by (c), and the length function of the coding is PRF. $\square$

**1.13 Def:** Let $f : \mathbb{N}^{n+1} \to \mathbb{N}$. Define $\bar{f} : \mathbb{N}^{n+1} \to \mathbb{N}$ by:

(a) $\bar{f}(0, x_1, \dots x_n) = 0 \; (= \langle \rangle)$

(b) $\bar{f}(k+1, x_1, \dots x_n) = \langle z_1, \dots z_m, f(k, x_1, \dots x_n) \rangle$ if $\bar{f}(k, x_1, \dots x_n) = \langle z_1, \dots z_m \rangle$

$= \langle f(0, \vec{x}), f(1, \vec{x}), \dots f(k, \vec{x}) \rangle$

(Wertverlaufsrekursion)

**1.14 Thm (Recursion Thm):** Let $g : \mathbb{N}^{n+2} \to \mathbb{N}$ be PRF and let $f : \mathbb{N}^{n+1} \to \mathbb{N}$ be with $f(n, \vec{x}) = g(n, \bar{f}(n, \vec{x}), \vec{x})$.

Then $f$ is uniquely defined and PRF.

**Proof:** $f$ PRF: 1.) $\bar{f}$ prim rec., since $\bar{f}(0, \vec{x}) = \langle \rangle$ and $\bar{f}(k+1, \vec{x}) = \bar{f}(k, \vec{x}) \frown \langle f(k, \vec{x}) \rangle$

$= \bar{f}(k, \vec{x}) \frown \langle g(k, \bar{f}(k, \vec{x}), \vec{x}) \rangle$

and $g$ prim rec., $\langle \rangle$ prim rec. (L 1.12), concatenation prim rec.

(uses decoding: $x \frown y = \langle (x)_0, \dots (x)_{length(x) \dot{-} 1}, (y)_0, \dots (y)_{length(y) \dot{-} 1} \rangle$)

2.) $\bar{f}$ prim rec. if and only $f$ prim rec., since:

"$\Rightarrow$" $\bar{f} \in PRF \Rightarrow f(n, \vec{x}) = (\bar{f}(n+1, \vec{x}))_n$ PRF

"$\Leftarrow$" $f \in PRF$, then $\bar{f} \in PRF$ since it is defined by recursion, coding PRF, concatenation PRF. $\square$

1.15 <u>Corollary</u>: Let $h_1, \dots h_m$ be regressive functions, i.e. $h_i(z,\vec{x}) < z \ \forall i$.
Let $f$ be with $f(z,\vec{x}) = g(f(h_1(z,\vec{x}),\vec{x}), \dots f(h_m(z,\vec{x}),\vec{x}), z, \vec{x})$.
If $g, h_1, \dots h_m \in PRF$, then $f \in PRF$.

1.16 <u>Remark</u>: (a) We may define primitive recursive operators $\mathcal{O}$ by
$\mathcal{O}(P_1, \dots P_n)$, $P_1, \dots P_n$ predicates and
$\chi_{\mathcal{O}(P_1 \dots P_n)}(\vec{x}) = g(\chi_{P_1}(\vec{x}), \dots \chi_{P_n}(\vec{x}), \vec{x})$, $g \in PRF$.

(b) We may define a simultaneous recursion by
$f_i(0,\vec{x}) = g_i(\vec{x})$, $f_i(k+1,\vec{x}) = h_i(k, f_1(k,\vec{x}), \dots f_m(k,\vec{x}), \vec{x})$
and prove $g_i, h_i \in PRF \Rightarrow f_i \in PRF$.

1.17 <u>Def.</u>: For $f \in PRT$ define its <u>Gödel number</u> $\ulcorner f \urcorner$ by:

("< no., input size, data >")

(i) $\ulcorner \underline{S} \urcorner := \langle 0, 1 \rangle$

(ii) $\ulcorner \underline{C}^n_k \urcorner := \langle 1, n, k \rangle$, $n \geq 1$

(iii) $\ulcorner \underline{P}^n_k \urcorner := \langle 2, n, k \rangle$, $1 \leq k \leq n$, $n \geq 1$

(iv) $\ulcorner \underline{Sub}(\underline{g}, \underline{h_1} \dots \underline{h_m}) \urcorner := \langle 3, (\ulcorner \underline{h_1} \urcorner)_1, \ulcorner \underline{g} \urcorner, \ulcorner \underline{h_1} \urcorner, \dots \ulcorner \underline{h_m} \urcorner \rangle$

(v) $\ulcorner \underline{Rec}(\underline{g}, \underline{h}) \urcorner := \langle 4, (\ulcorner g \urcorner)_1 + 1, \ulcorner \underline{g} \urcorner, \ulcorner \underline{h} \urcorner \rangle$

Put $PRI := \{x \in \mathbb{N} \mid \exists \underline{f} \in PRT : x = \ulcorner \underline{f} \urcorner\}$

16.4. 1.18 <u>Lemma</u>: $PRI \in PRP$.

<u>Proof</u>: $x \in PRI \iff Seq(x) \wedge x \neq 0 \wedge$

$$
\begin{aligned}
\Big[ & ((x)_0 = 0 \wedge length(x) = 2 \wedge (x)_1 = 1) \\
\vee\ & ((x)_0 = 1 \wedge length(x) = 3 \wedge (x)_1 \neq 0) \\
\vee\ & ((x)_0 = 2 \wedge length(x) = 3 \wedge (x)_1 \neq 0 \wedge 1 \leq (x)_2 \leq (x)_1) \\
\vee\ & ((x)_0 = 3 \wedge length(x) = ((x)_2)_1 + 3 \\
& \wedge (\forall i < length(x) : i < 2 \vee (x)_i \in PRI) \\
& \wedge (\forall i < length(x) : i < 3 \vee ((x)_i)_1 = (x)_1)\ ) \\
\vee\ & \dots
\end{aligned}
$$

Then Thm 1.14 for $\chi_P \in PRF$. (for "$(x)_i \in PRI$")

$\square$

1.19 <u>Def.</u>: Let $e \in PRI$ and let $\underline{f} \in PRT$ such that $e = \ulcorner \underline{f} \urcorner$.

Put $\underline{[e]} := \underline{f}$ and $[e] := val(\underline{[e]})$ the <u>prim. rec. index</u>.

1.20 <u>Thm</u>: For any $f \in PRF^n$, we find $e \in PRI$ such that $f(\vec{x}) = \Phi^n(e, \vec{x})$,

where $\Phi^n(e, \vec{x}) := \begin{cases} [e](\vec{x}) & \text{if } e \in PRI \text{ and } (e)_1 = n \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u>: $f \in PRF^n \Rightarrow \exists \underline{f} \in PRT^n : f = val(\underline{f})$. Put $e := \ulcorner \underline{f} \urcorner \in PRI$.

Then $\Phi^n(e, \vec{x}) = [e](\vec{x}) = val(\underline{f})(\vec{x}) = f(\vec{x})$. $\square$

1.21 <u>Remark</u>: $\Phi^n \notin PRF$.

Indeed, assume $\Phi^n \in PRF$ and put $h(x_1, \ldots x_n) := \Phi^n(x_1, x_1, \ldots x_n) + 1$.

Then $h \in PRF$, i.e. we find $e \in PRI$ with $h(\vec{x}) = \Phi^n(e, \vec{x})$.

Thus $\Phi^n(e, e, x_2, \ldots x_n) = h(e, x_2, \ldots x_n) = \Phi^n(e, e, x_2, \ldots x_n) + 1$ $\lightning$ $\square$