# § 2 Partially recursive functions and the arithmetic hierarchy

**2.1 Def.:** The class PT of <u>partially recursive terms</u> (also: <u>$\mu$-recursive terms</u>) is defined by:

(i)–(v): $\underline{S}$, $\underline{C}^n_k$, $\underline{P}^n_k$, $\underline{Sub}(\underline{h}, \underline{g}_1, \ldots \underline{g}_m)$, $\underline{Rec}(\underline{g}, \underline{h}) \in PT$ (see Def. 1.1)

(vi) Given $\underline{h}$ $(n+1)$-ary PT, then $\underline{\mu}\,\underline{h}$ is an $n$-ary PT.

**2.2 Def.:** (a) A <u>partial function</u> $f: \mathbb{N}^n \to_p \mathbb{N}$ is a function with $\mathrm{dom}\, f \subseteq \mathbb{N}^n$.
A partial function is <u>total</u>, if $\mathrm{dom}\, f = \mathbb{N}^n$.

(b) Given $\underline{f} \in PT^n$, we define $\mathrm{val}(\underline{f}): \mathbb{N}^n \to_p \mathbb{N}$ by:

(i)–(v) $\mathrm{val}(\underline{S})$, $\mathrm{val}(\underline{C}^n_k)$, $\mathrm{val}(\underline{P}^n_k)$, $\mathrm{val}(\underline{Sub}(\underline{h}, \underline{g}_1, \ldots \underline{g}_m))$,
$\mathrm{val}(\underline{Rec}(\underline{g}, \underline{h}))$ as in Def. 1.2.

(vi) $\mathrm{val}(\underline{\mu}\,\underline{h})(x_1, \ldots x_n) :\simeq \min\{k \in \mathbb{N} \mid \mathrm{val}(\underline{h})(k, x_1, \ldots x_n) \simeq 0 \wedge \forall l \leq k\ \exists z \neq 0 : \mathrm{val}(\underline{h})(l, x_1, \ldots x_n) \simeq z\}$

smallest root ("Nullstelle")

is only defined, if $\{\ldots\} \neq \emptyset$.

**2.4 Remark:** (a) $PRF \subsetneqq \mathbb{P}$, Ackermann $\in \mathbb{P}$, total function, Ackermann $\notin PRF$.

(b) busy beaver (= Rado function) $\notin \mathbb{P}$.

**2.3 Def.:** A partial function $f: \mathbb{N}^n \to_p \mathbb{N}$ is <u>$\mu$-partially recursive</u>, if there is $\underline{f} \in PT^n$ with $f = \mathrm{val}(\underline{f})$. $\mathbb{P} := \{\mu\text{-partially recursive functions}\}$. $\mathbb{F} := \{f \in \mathbb{P}\ \text{total}\}$.

**2.5 Theorem:** $\mathbb{P}$ is the smallest class containing the functions $S$, $C^n_k$, $P^n_k$ and being closed under Sub, Rec and unbounded search for roots. [illegible].

*Proof:* $\mathbb{P}$ satisfies these properties by definition; and for any $f \in \mathbb{P}$, we have $f \in \mathcal{P}$, where $\mathcal{P}$ is the smallest such class, since $f = \mathrm{val}(\underline{f})$ and $\mathrm{val}(\underline{S}), \mathrm{val}(\underline{C}^n_k), \ldots \in \mathcal{P}$. $\square$

**2.6 Def.:** Let $P \subseteq \mathbb{N}^{n+1}$ be a predicate. Then $\mu P(x_1, \ldots x_n) \simeq \min\{k \in \mathbb{N} \mid P(k, x_1, \ldots x_n)\}$.

**2.7 Lemma:** $P \in PRP \Rightarrow \mu P \in \mathbb{P}$.

*Proof:* $P \in PRP \overset{1.8}{\Longrightarrow} \chi_P \in PRF \subseteq \mathbb{P}$. And $\mu P \simeq \mu\, \overline{sg} \circ \chi_P$, where $\overline{sg}(0) = 1$, $\overline{sg}(n+1) = 0$, $\overline{sg} \in PRF$. $\square$

$\rightsquigarrow$ (bounded search, 1.10)
Ex. 1

2.8 <u>Remark</u>: Functions in $\mathbb{P}$ are "intuitively" computable, i.e. we have an algorithm for them – which might not terminate. For total functions in $\mathbb{P}$, however, it does terminate. The class $\mathbb{P}$ coincides with the following other classes from computational complexity theory:

- The class of all register machine computable functions (like Turing machine).
- WHILE computability
- GOTO computability
- RAM computability
- $\lambda$-calculus

The link for the famous class $P = PTIME$ (from "$P$ vs $NP$") is

$P := \{$ problems which may be solved by deterministic Turing machines in polynomial time$\}$

$NP := \{$ problems, which may be solved by non-determ. Turing machines in polynomial time$\}$

with $P \subseteq NP \subseteq \mathbb{P}$. (Open, 1,000,000 \$: $P = NP$ ?)

I think?

2.9 <u>Def</u>: For $f \in PT$ define its <u>Gödel number</u> $\ulcorner f \urcorner$ by:

(i) – (v) see Def. 1.17

(vi) $\ulcorner \mu f \urcorner := \langle 5, (\ulcorner f \urcorner)_1 \dot- 1, \ulcorner f \urcorner \rangle$

$PI := \{ e \in \mathbb{N} \mid \exists f \in PT : e = \ulcorner f \urcorner \}$, $\underline{\{e\}} := f$ for $e = \ulcorner f \urcorner \in PI$.

2.10 <u>Lemma</u>: $PI \in PRP$.

<u>Proof</u>: As in Lemma 1.18. □

2.11 <u>Def</u>: The <u>computability predicate</u> $B$ is defined as

$B(\langle e, x, z, b \rangle) :\Longleftrightarrow b$ encodes a computation of the term $\underline{\{e\}}((x)_0, \ldots (x)_{\mathrm{length}(x) \dot- 1})$ with result $z$

Hence:

$B(\langle \ulcorner \underline{S} \urcorner, \langle x_1 \rangle, x_1 + 1, \langle \rangle \rangle)$

$B(\langle \ulcorner \underline{C_k^n} \urcorner, \langle x_1, \ldots x_n \rangle, k, \langle \rangle \rangle)$

$B(\langle \ulcorner \underline{P_k^n} \urcorner, \langle x_1, \ldots x_n \rangle, x_k, \langle \rangle \rangle)$

$B(\langle \ulcorner \underline{\mathrm{Sub}}(g, h_1, \ldots h_m) \urcorner, x, y, \langle b_{h_1}, \ldots b_{h_m}, b_g \rangle \rangle)$

if $b_{h_i} = \langle \ulcorner \underline{h_i} \urcorner, x, u_i, b'_{h_i} \rangle \in B$

$b_g = \langle \ulcorner g \urcorner, \langle u_1, \ldots u_m \rangle, y, b'_g \rangle \in B$

**2.12 Lemma:** $\mathcal{B} \in PRP$

Proof: similar to Lemma 1.18, using the Recursion Thm 1.14. □

**2.13 Normal Form Theorem of Kleene:** "universal predicate"

There is $T^n \in PRP^{n+2}$ and $U \in PRF$ such that for any $f \in \mathbb{P}^n$ we find a number $e \in \mathbb{N}$ with $f(x_1, \ldots x_n) \simeq U(\mu y\, T^n(e, x_1, \ldots x_n, y))$.

Proof: $T^n(e, x_1 \ldots x_n, w) :\iff \mathcal{B}(w) \wedge (w)_0 = e,\ (w)_1 = \langle x_1 \ldots x_n \rangle$

$U(w) := (w)_2$

$e := \ulcorner \underline{f} \urcorner$ for $\underline{f} \in \mathbb{P}^n$ with $val(\underline{f}) = f$. Hence $\{\underline{e}\} = \underline{f}$

Case 1: $(x_1, \ldots x_n) \in \mathrm{dom}\, f$.

Then $\{\underline{e}\}\, \underline{x_1} \ldots \underline{x_n}$ has a result $z$, i.e. there is a $b$ such that $w := \langle e, \langle x_1 \ldots x_n \rangle, z, b \rangle \in \mathcal{B}$, $(w)_0 = e$, $(w)_1 = \langle x_1 \ldots x_n \rangle$

$\Rightarrow U(\mu y\, T^n(e, x_1 \ldots x_n, y)) \underset{(y=w)}{=} (w)_2 = z = f(x_1 \ldots x_n)$.

Case 2: $(x_1 \ldots x_n) \notin \mathrm{dom}\, f$. Then no such $b$, no such $w$ exists and $U(\mu y\, T^n(e, x_1 \ldots x_n, y))$ is undefined, just like $f(x_1 \ldots x_n)$. □

(2nd Recursion Thm)

**2.14 Corollary:** If we put $\Phi^n(e, \vec{x}) \simeq \{e\}(\vec{x}) \simeq U(\mu y\, T^n(e, x_1, \ldots x_n, y))$, then $\Phi^n \in \mathbb{P}$. It is universal, i.e. $\forall f \in \mathbb{P}^n\ \exists e \in \mathbb{N} : f(\vec{x}) \simeq \Phi^n(e, \vec{x})$.
(*)

Proof: $U(\mu y T^n) \in \mathbb{P}$. □

**2.15 Remark:** Recall Remark 1.21: $\Phi^n \notin PRF$. The argument was:

($A$: $\Phi^n \in PRF \Rightarrow$) for $h(\vec{x}) :\simeq \Phi^n(x_1, \vec{x}) + 1$ find $e \in \mathbb{N}$ with $h(\vec{x}) \simeq \Phi^n(e, \vec{x})$.

Then $\Phi^n(e, e, x_2, \ldots x_n) \simeq h(e, x_2, \ldots x_n) \simeq \Phi^n(e, e, x_2, \ldots x_n) + 1$

$\Rightarrow e \notin \mathrm{dom}\, h$ ! (no ↯)

27.4.

**2.16 Def.:**

(a) $P \subseteq \mathbb{N}^n$ <u>recursive</u> $:\iff \chi_P \in \mathbb{F}$. "have an algorithm" = "(totally) computable", decidable

| | $n \in P$ | $n \notin P$ |
|---|---|---|
| (a) | ✓ | ✓ |
| (b) | ✓ | ? |

(b) $P \subseteq \mathbb{N}^n$ <u>recursively enumerable</u> $:\iff \exists f : \mathbb{N} \to \mathbb{N}^n,\ f_1, \ldots f_n \in \mathbb{F}$: $P = rg(f)$. "can decide $\vec{x} \in P$ if this is the case"

**2.17 Lemma:** $P$ recursive $\Rightarrow \neg P$ rec.

Proof: $\chi_{\neg P} = \overline{sg} \circ \chi_P$, $\overline{sg} \in PRF \subseteq \mathbb{F}$. □

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(*) We even have: $\forall f \in \mathbb{P}^{n+1}\ \exists e \in P\underline{I} : \{e\}(\vec{x}) \simeq f(e, \vec{x})$
(2nd Recursion Thm)

**2.18 Thm:** $P \subseteq \mathbb{N}^n$ rec. enumer. $\Longleftrightarrow$ $\exists R \subseteq \mathbb{N}^{n+1}$ rec. with

$$P(\vec{x}) \Leftrightarrow \exists y : R(\vec{x}, y)$$

**Proof:** „$\Rightarrow$" if $P \neq \emptyset$, then $P = rg(f)$, $f \in \mathbb{F}$. Hence

$$P(\vec{x}) \Leftrightarrow \exists y : f(y) = \vec{x}$$

„$\Leftarrow$" if $P \neq \emptyset$, choose $\vec{x}_0 \in P$.

Put $f(x) :\simeq \begin{cases} ((x)_0, \ldots (x)_{length(x) \dot{-} 2}) & \text{if } ((x)_0, \ldots (x)_{l(x) \dot{-} 1}) \in R \\ \vec{x}_0 & \text{otherwise} \end{cases}$

Then $P = rg(f)$. $\square$

**2.19 Cor:** $P$ rec. enumer. $\Leftarrow$ $P$ rec. $(Px \Leftrightarrow \exists y : y = y \wedge Px)$

**2.20 Thm:** The class of rec. enumer. predicates is not closed under taking complements and $\forall$ quantifications.

**Proof:** 1.) A: $P$ rec. enum. $\Rightarrow$ $\neg P$ rec. enum.

Put $K := \{x \mid \exists y : T^n(x, x, y)\}$ rec. enum. $\overset{A}{\Rightarrow}$ $\neg K$ rec. enum.

$\underset{\text{Sheet 2}}{\Rightarrow}$ $\exists e : \neg K = W_e$ for $W_e^n := \{\vec{x} \mid \underbrace{\exists y : T^n(e, \vec{x}, y)}_{\Leftrightarrow\ \vec{x} \in dom\{e\} = \neg K}\}$

But then $e \in W_e \Leftrightarrow e \in \neg K \Leftrightarrow e \notin K \Leftrightarrow e \notin W_e$ ↯

2.) $P$ rec. enum. $\overset{2.18}{\Rightarrow}$ $(P\vec{x} \Leftrightarrow \exists y : R(\vec{x}, y))$, $R$ rec.

$\Rightarrow$ $\neg R$ rec. $\overset{2.19}{\Rightarrow}$ $\neg R$ rec. enum.

A: $\underbrace{\forall y : \neg R(\vec{x}, y)}_{\Leftrightarrow\ \neg(\exists y : R(\vec{x}, y)) \Leftrightarrow \neg P\vec{x}}$ rec. enum. ↯ (1.) $\square$

**2.21 Thm:** $P$ rec. $\Leftrightarrow$ $P$ rec. enum. $\wedge$ $\neg P$ rec. enum.

„$\Rightarrow$" 2.19 & 2.17.

„$\Leftarrow$" $P\vec{x} \overset{2.18}{\Leftrightarrow} \exists y : R_1(\vec{x}, y)$, $\neg P\vec{x} \overset{2.18}{\Leftrightarrow} \exists y : R_2(\vec{x}, y)$

$f(\vec{x}) :\simeq \mu y[R_1(\vec{x}, (y)_0) \vee R_2(\vec{x}, (y)_1)]$, $f \in \mathbb{F}$.

$P\vec{x} \Leftrightarrow R_1(\vec{x}, (f(\vec{x}))_0)$ rec. (since $\mathbb{F}$ is closed under Sub) $\square$

**2.22 Cor:** $P$ rec. enum. $\not\Rightarrow$ $P$ rec.

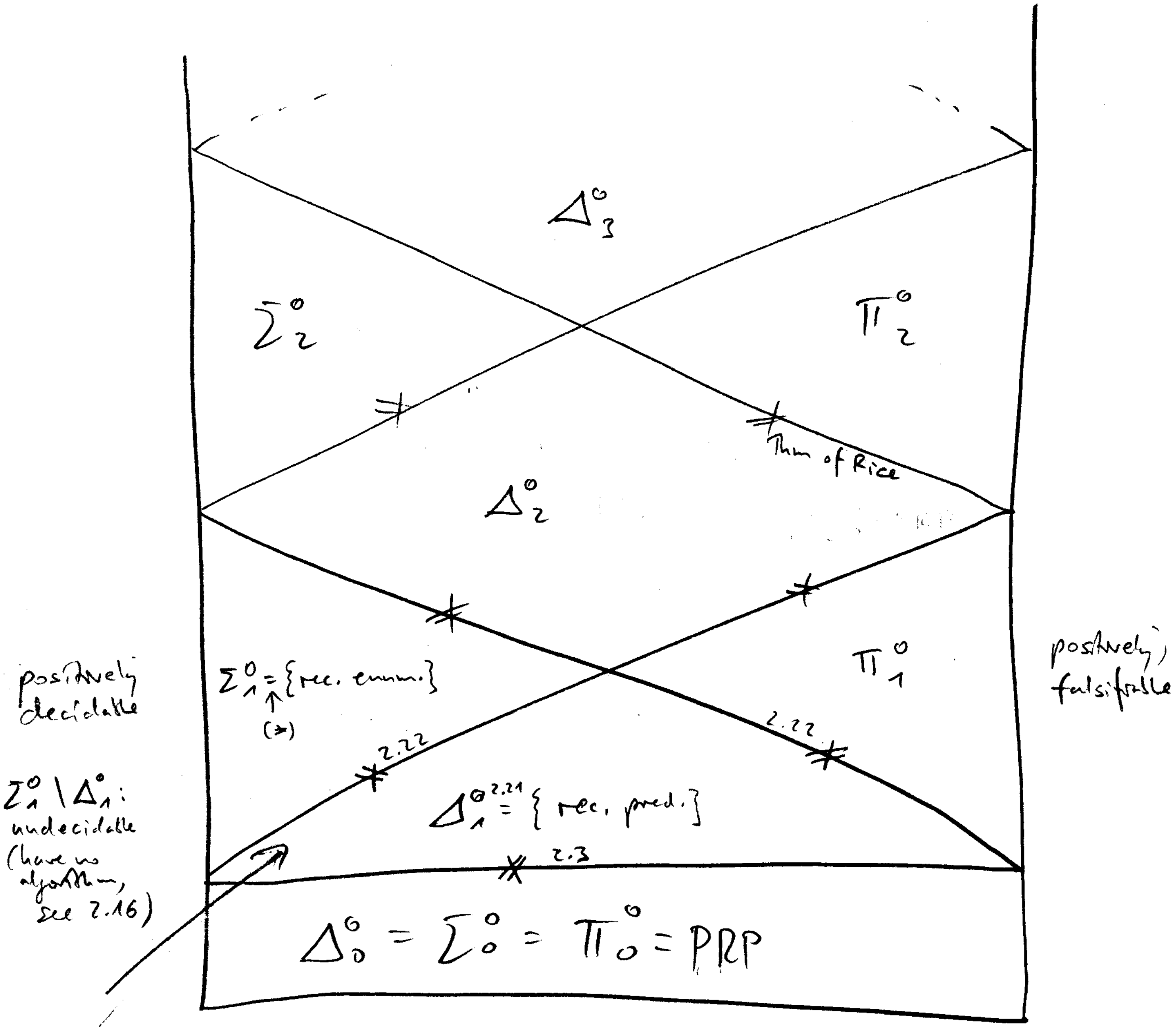
**Proof:** The halting problem $K$ from 2.20. $\square$

**2.23 Def.:** $\Delta^0_0 := \Pi^0_0 := \Sigma^0_0 :=$ PRP.

$\Sigma^0_{n+1} := \{ P \mid \exists R \in \Pi^0_n : (P\vec{x} \Leftrightarrow \exists y : R(\vec{x}, y)) \}$

$\Pi^0_{n+1} := \neg \Sigma^0_{n+1}$

$\Delta^0_{n+1} := \Pi^0_{n+1} \cap \Sigma^0_{n+1}$




$x \in P$ decidable ($A_1$)
$x \notin P$ decidable ($A_2$)
$\Rightarrow P$ decidable
(alg. $A_1 A_2 A_1 A_2 \ldots$)

(∗) $P \in \Sigma^0_1 \overset{2.19}{\Longleftrightarrow} (P\vec{x} \Leftrightarrow \exists y : \underbrace{T^n(e, \vec{x}, y)}_{\text{PRP}})$

Note: $\underset{\Sigma^0_n}{P\vec{x}} \ldots \Leftrightarrow \exists y_1 \forall y_2 \exists y_3 \ldots \underset{\Delta^0_0}{R}(\vec{x}, y_n \ldots, y_2, y_1)$