Ai–: Statt– in S.4

§4 PL1 (first order logic)

4.1 Def.: (Syntax)

a) A PL1 language $\mathcal{L}(\mathcal{C},\mathcal{F},\mathcal{P})$ is given by

(i) the logical symbols:

- free variables $u, v, v_1, \ldots$
- bound variables

} countably many

- the logical connectives $\neg, \wedge, \vee, \to$
- the quantifiers $\forall, \exists$

} the same for all PL1 languages

(ii) the nonlogical symbols:

- $\mathcal{C}$, a set of symbols for constants
- $\mathcal{F}$, a set of symbols for functions
- $\mathcal{P}$, a set of symbols for predicates

} called the signature

b) The terms in $\mathcal{L}(\mathcal{C},\mathcal{F},\mathcal{P})$ are

- all free variables $u, v, v_1, \ldots$ — we write $FV(u_i)=\{u_i\}$
- all constant symbols $c \in \mathcal{C}$; $FV(c)=\emptyset$
- $t_1 \ldots t_n$ ~~terms~~, $f \in \mathcal{F}$, $\# f = n \Rightarrow f t_1 \ldots t_n$ term, $FV(f t_1 \ldots t_n) = \bigcup_{i=1}^{n} FV(t_i)$

Here, $FV(t)$ are the free variables of t. If $FV(t)=\emptyset$, then t is called closed.

c) The formulas in $\mathcal{L}(\mathcal{C},\mathcal{F},\mathcal{P})$ are

- $t_1 \ldots t_n$ ~~terms~~, $P \in \mathcal{P}$, $\# P = n \Rightarrow P t_1 \ldots t_n$ "atom formula"
  $FV(P t_1 \ldots t_n) = \bigcup FV(t_i)$
  $BV(P t_1 \ldots t_n) = \emptyset$

- A, B formulas $\Rightarrow$ $\neg A$, $A \wedge B$, $A \vee B$, $A \to B$ formulas

  $FV(\neg A) = FV(A)$, $BV(\neg A) = BV(A)$
  $FV(A \wedge B) = FV(A \vee B) = FV(A \to B) = FV(A) \cup FV(B)$
  $BV(A \wedge B) = BV(A \vee B) = BV(A \to B) = BV(A) \cup BV(B)$

- A formula, x bounded variable, $x \notin BV(A)$, $u \in FV(A)$
  $\Rightarrow \forall x\, A_u(x)$, $\exists x\, A_u(x)$ formulas
  where each occurrence of the free variable u in A is replaced by x in order to obtain $A_u(x)$
  $FV(Qx\, A_u(x)) = FV(A) \setminus \{u\}$
  $BV(Qx\, A_u(x)) = BV(A) \cup \{x\}$

Here, $BV(B)$ are the bound variables of B. If $FV(B)=\emptyset$, then B is a theorem.

4.2 Def: (Semantics) A $\underline{\mathcal{L}(\mathcal{C},\mathcal{F},\mathcal{P})\text{-structure}}$ $\mathcal{S}=(S,\mathcal{C}^{\mathcal{S}},\mathcal{F}^{\mathcal{S}},\mathcal{P}^{\mathcal{S}})$ consists in

- the <u>support</u> $S\neq\emptyset$
- $\mathcal{C}^{\mathcal{S}}:\mathcal{C}\to S$, $c\mapsto c^{\mathcal{S}}$ interpretation of constants
- $\mathcal{F}^{\mathcal{S}}:\mathcal{F}\to\bigcup_{n\in\mathbb{N}}\{f:S^n\to S\}$, $f^{\mathcal{S}}:S^{\#f}\to S$ interpretation of functions
- $\mathcal{P}^{\mathcal{S}}:\mathcal{P}\to\bigcup_{n\in\mathbb{N}}\mathcal{P}(S^n)$, $P^{\mathcal{S}}\subseteq S^{\#P}$ interpretation of predicates

4.3 Examples: If $\mathcal{L}=(\{1\},\{\circ\},\{=\})$ is the language of group theory, i.e.

$\forall x\,(\circ(x,e)=x \wedge \circ(e,x)=x)$ & $\forall x\exists y\,(\circ(x,y)=e)$
$\wedge\forall x\exists y\,(\circ(y,x)=e)$

Then $\mathbb{Z}=(\mathbb{Z},\{0\},\{+\},\{=\})$ and $\mathbb{Q}=(\mathbb{Q},\{1\},\{\cdot\},\{=\})$ are $\mathcal{L}$-structures.

4.4 Def: Given $\mathcal{L}$ and $\mathcal{S}$, an <u>interpretation</u> is a map $\Phi: FV\to S$.

a) • $u^{\mathcal{S}}[\Phi]:=\Phi(u)$ interpretation of terms

- $c^{\mathcal{S}}[\Phi]:=c^{\mathcal{S}}$
- $(ft_1\ldots t_n)^{\mathcal{S}}[\Phi]:=f^{\mathcal{S}}(t_1^{\mathcal{S}}[\Phi],\ldots t_n^{\mathcal{S}}[\Phi])$

b) • $(Pt_1\ldots t_n)^{\mathcal{S}}[\Phi]:=\begin{cases}\text{true} & \text{if } P^{\mathcal{S}}(t_1^{\mathcal{S}}[\Phi],\ldots t_n^{\mathcal{S}}[\Phi])\\ \text{false} & \text{otherwise}\end{cases}$ interpretation of formulas

- $(\neg A)^{\mathcal{S}}[\Phi]:=\neg(A^{\mathcal{S}}[\Phi])$, $(A\circ B)^{\mathcal{S}}[\Phi]:=A^{\mathcal{S}}[\Phi]\circ B^{\mathcal{S}}[\Phi]$ for $\circ\in\{\wedge,\vee,\to\}$
- $(Qx\,Fu(x))^{\mathcal{S}}[\Phi]:=\begin{cases}\text{true} & \text{if } (Fu(\underline{x}))^{\mathcal{S}}[\Phi]=\text{true } Qx\in S,\ \underline{x}^{\mathcal{S}}=x\\ \text{false} & \text{otherwise}\end{cases}$

We write $\mathcal{S}\models F[\Phi]$ instead of $F^{\mathcal{S}}[\Phi]=$ true
and $\mathcal{S}\not\models F[\Phi]$ instead of $F^{\mathcal{S}}[\Phi]=$ false

"The structure $\mathcal{S}$ satisfies / does not satisfy the formula F given the interpretation Φ"

4.5 Def: Let M be a set of $\mathcal{L}$-formulas.

a) M is <u>consistent</u>, if there is an $\mathcal{L}$-structure $\mathcal{S}$ and an interpretation Φ such that $\mathcal{S}\models F[\Phi]$ $\forall F\in M$

b) M is <u>valid</u> in $\mathcal{S}$, if $\mathcal{S}\models F[\Phi]$ $\forall F\in M$ $\forall\Phi$ interpretations

c) M is <u>universally valid</u> ($\models M$), if M is valid in all $\mathcal{L}$-structures $\mathcal{S}$.

4.6 <u>Remark</u>: a) We may define <u>extensions</u> of languages by adding new constants.

b) We may define equivalence on a semantics level by asking for $F^S[\Phi] = G^S[\Phi] \quad \forall S \; \forall \Phi$

c) We may define <u>Boolean interpretations</u> by defining $\Phi_B : PA \to \{true, false\}$ only on the formulas (inductively as in 4.4 b); atoms $\mapsto$ true or false...). Note that any interpretation Φ as in Def. 4.4 induces a Boolean interpretation Φ_B^S with $\Phi_B^S(F) = F^S[\Phi]$ for all formulas F.
We may then define $\models_b \mathcal{M}$, if $\Phi_B(F) = true \quad \forall F \in \mathcal{M} \; \forall \Phi_B$
In general $\models_b F \underset{\not\Leftarrow}{\Rightarrow} \models F$. Since not all Φ_B come from Φ_B^S.
[Idea: „$\Rightarrow$" use Φ_B^S „$\Leftarrow$" $F :=$ "$x = x$" is an atom which may be set to false.]

d) There is a Compactness Theorem in Boolean logic:
"Boolean consistent": $\exists \Phi_B \; \forall F \in \mathcal{M} : \Phi_B(F) = true \iff \forall \mathcal{M}_0 \subseteq_{fin} \mathcal{M} \; \exists \Phi_B \; \forall F \in \mathcal{M}_0 : \Phi_B(F) = true$

e) $\mathcal{M}$ is a <u>Henkin set</u>, if $\mathcal{M}$ contains all formulas of the form $F_u(t) \to \exists x F_u(x)$ as well as for any $\exists x F_u(x) \in \mathcal{M}$ also $\exists x F_u(x) \to F_u(c)$ for a constant $c = c_{\exists x F_u(x)}$.
For Henkin sets $\mathcal{M}$, we have:

$$\mathcal{M} \text{ consistent} \iff \mathcal{M} \text{ Boolean consistent}$$

f) There is a Compactness Theorem for PL1:

$$\mathcal{M} \text{ consistent} \iff \forall \mathcal{M}_0 \subseteq_{fin} \mathcal{M} : \mathcal{M}_0 \text{ consistent}$$

[<u>Proof</u>: „$\Rightarrow$" trivial „$\Leftarrow$" Use a Henkin extension of $\mathcal{M}$ and (d).

g) We may define $\mathcal{M} \models_b F \;:\iff \forall \Phi_B : \left([\forall G \in \mathcal{M} : \Phi_B(G) = true] \Rightarrow \Phi_B(F) = true \right)$

4.7 Defin.: $\mathcal{M} \models F :\Longleftrightarrow \forall \mathfrak{S}\, \forall \Phi : [\mathfrak{S} \models \mathcal{M}[\Phi] \Rightarrow \mathfrak{S} \models F[\Phi]]$

set of formulas (for $\mathcal{M}$); i.e. $\mathfrak{S} \models G[\Phi]\ \forall G \in \mathcal{M}$ (for $\mathfrak{S} \models \mathcal{M}[\Phi]$)

4.8 Theorem: The procedure $\models$ admits the following logical conclusions for sets $\mathcal{M}, \mathcal{N}$ of formulas:

(a) $\mathcal{M} \models F \Longleftrightarrow \mathcal{M} \cup \{\neg F\}$ inconsistent

[Proof: $\mathcal{M} \cup \{\neg F\}$ consistent $\Longleftrightarrow \exists \mathfrak{S}, \Phi : \mathfrak{S} \models \mathcal{M}[\Phi] \wedge \mathfrak{S} \models \neg F[\Phi] \Longleftrightarrow \mathcal{M} \not\models F$]

(b) $\mathcal{M}$ inconsistent $\Longrightarrow \mathcal{M} \models F\ \forall F$ [Use (a).]

(c) $\mathcal{M} \models F \Longrightarrow \exists \mathcal{M}_0 \underset{\text{fin}}{\subseteq} \mathcal{M} : \mathcal{M}_0 \models F$ [Use (a) and apply Rem. 4.6(f).]

(d) $\mathcal{M} \cup \{F\} \models G \Longleftrightarrow \mathcal{M} \models \{F \to G\}$

[Proof: $\mathcal{M} \cup \{F, \neg G\}$ incons. $\Longleftrightarrow \mathcal{M} \cup \{F \to G\}$ incons.]

(e) $\mathcal{M} \subseteq \mathcal{N},\ \mathcal{M} \models F \Longrightarrow \mathcal{N} \models F$ [Use (a).]

(f) $\models F \Longleftrightarrow \mathcal{M} \models F\ \forall \mathcal{M}$ [$\underset{\text{"}}{\text{"}}\{\neg F\}$ incons.$\underset{\text{"}}{\text{"}} \Longleftarrow$ "$\mathcal{M} = \emptyset$"]

(g) $F \in \mathcal{M} \Longrightarrow \mathcal{M} \models F$ [$\{F, \neg F\}$ incons.]

(h) $\mathcal{M} \models_{\mathfrak{S}} \mathcal{N},\ \mathcal{N} \models F \Longrightarrow \mathcal{M} \models F$

$\mathcal{M} \models \mathcal{N},\ \mathcal{N} \models_{\mathfrak{S}} F \Longrightarrow \mathcal{M} \models F$

[Proof: Let $\mathfrak{S}, \Phi$ be given, $\mathfrak{S} \models \mathcal{M}[\Phi]$. Then $\Phi^{\mathfrak{S}}_{\mathfrak{B}}(\mathcal{M}) = \text{true}$.

$\xRightarrow{\mathcal{M} \models_{\mathfrak{S}} \mathcal{N}} \Phi^{\mathfrak{S}}_{\mathfrak{B}}(\mathcal{N}) = \text{true} \Rightarrow \mathfrak{S} \models \mathcal{N}[\Phi] \xRightarrow{\mathcal{N} \models F} \mathfrak{S} \models F[\Phi]$.

Similarly for the second statement.]

we know that $\Phi^{\mathfrak{S}}_{\mathfrak{B}}$ comes from $\mathfrak{S}$!