# §5 Logical calculus

5.1 Def.: We define $M \vdash F$ inductively, $M$ a set of formulas,
(Hilbert calculus) $F$ a formula.

(i) $F \in M \Rightarrow M \vdash F$

(ii) $\models_B F \Rightarrow M \vdash F$

(iii) $M \vdash (F_u(t) \to \exists x\, F_u(x))$ & $M \vdash (\forall x\, F_u(x) \to F_u(t))$

(iv) $M \vdash A,\ M \vdash (A \to F) \Rightarrow M \vdash F$

(v) $M \vdash (F \to A),\ u \notin FV(M) \cup FV(A) \Rightarrow M \vdash (\exists x\, F_u(x) \to A)$
$M \vdash (A \to F),\ u \notin FV(M) \cup FV(A) \Rightarrow M \vdash (A \to \forall x\, F_u(x))$

5.2 Correctness Thm: $M \vdash F \Rightarrow M \models F$

Proof: (i) 4.8(g); (ii) $\models_B F \overset{4.6c}{\Rightarrow} \models F$, 4.8(f); (iii) Show $\models F_u(t) \to \exists x F_u(x)$
by proving for $\mathfrak{S}, \Phi$ given: $\mathfrak{S} \models F_u(t)[\Phi] \Rightarrow \mathfrak{S} \models \exists x\, F_u(x)[\Phi]$.
Then 4.8(f).
(iv) By induction hypothesis $M \models A$, $M \models (A \to F)$.
Also $\{A, A \to F\} \models_B F$. Then $M \models F$ by 4.8h.
(v) $M \models (F \to A), u \notin FV(M) \cup F(A)$ implies $M \models \exists x F_u(x) \to A$ using $\mathfrak{S}, \Phi$.
□

5.3 Completeness Thm: $M \models F \Rightarrow M \vdash F$

Proof: $M \models F \overset{4.8a}{\Rightarrow} M \cup \{\neg F\}$ inconsistent
$\Rightarrow M \cup \{\neg F\} \cup H_\mathcal{L}$ inconsistent, $H_\mathcal{L}$ some Henkin extension
$\overset{4.6e}{\Rightarrow} M \cup \{\neg F\} \cup H_\mathcal{L}$ Boolean inconsistent
$\overset{4.6d}{\Rightarrow} \exists M_0 \subseteq_{fin} M \cup \{\neg F\} \cup H_\mathcal{L}$ Boolean inconsistent
$\Rightarrow \models_B \neg A_1 \vee \dots \vee \neg A_m \vee \neg B_1 \vee \dots \vee \neg B_n \vee F$, $A_i \in M, B_i \in H_\mathcal{L}$
$\overset{5.1ii}{\Rightarrow} M \vdash \neg A_1 \vee \dots \vee \neg A_m \vee \neg B_1 \vee \dots \vee \neg B_n \vee F$
$\overset{5.1i\ \&\ iv}{\Rightarrow} M \vdash \neg B_1 \vee \dots \vee \neg B_n \vee F$
$\overset{\text{Henkin}\ \&\ 5.1iii}{\Rightarrow} M \vdash F$
□

5.4 Corollary: $\models F \Leftrightarrow \vdash F$ ("There is a correct and complete calculus for PL1")
Gödel: Every statement which is true by logical reasons may be deduced already by logical calculus (i.e. by a "proof").

5.5 Remark: Let $\mathcal{L}$ be a countable PL1 language. We may gödelize the formulas $F$ to $\ulcorner F \urcorner \in \mathbb{N}$ such that $F$ may be reconstructed from $\ulcorner F \urcorner$.

5.6 Theorem: $\{\ulcorner F \urcorner \mid \vDash F\} \in \Sigma^0_1 \setminus \Delta^0_1$.

Hence "proving" is undecidable.

Proof: We have $\vDash F \overset{4.8f}{\iff} \forall \mathcal{M}: \mathcal{M} \vDash F \overset{5.2\&5.3}{\iff} \forall \mathcal{M}: \mathcal{M} \vdash F$

$\Sigma^0_1$: Write $\mathcal{M} \vdash F \iff \exists s \in Seq \wedge \forall i < length(s): (s)_i$ axiom or ...

Then 2.18.

$\neg\Delta^0_1$: As in Thm. 3.3, construct a Turing machine with $dom(Machine) \in \Sigma^0_1 \setminus \Delta^0_1$. Then construct a formula $F_0$ such that $\vDash F_0 u_1 \dots u_n \iff (u_1, \dots u_n) \in dom(Machine)$. $\square$

5.7 Remark: (a) There are a number of alternative calculi, for instance the Tait calculus $\vdash_T$: Here, the language is given by free and bound variables, $\wedge, \vee$ and $\exists, \forall$ (so no $\neg$, no $\rightarrow$); the nonlogical symbols are $\mathcal{C}, \mathcal{F}, \mathcal{P}$ and $\overline{\mathcal{P}} := \{\overline{P} \mid P \in \mathcal{P}\}$, interpreted as $\overline{P}^{\mathcal{S}} := \{(s_1, \dots s_n) \mid (s_1, \dots s_n) \notin P^{\mathcal{S}}\}$. For a formula $F$ define $\sim F$ in the sense of $\neg F$. Then define a Tait calculus by "$\vdash_T \mathcal{M}, A_0$ and $\vdash_T \mathcal{M}, A_1 \Rightarrow \vdash_T \mathcal{M}, (A_0 \wedge A_1)$" etc. This calculus is correct and complete, i.e. $\mathcal{M} \vdash_T F \iff \mathcal{M} \vDash F$.
"$\Rightarrow$" "$\Leftarrow$"

(b) For other languages (and their logic "$\mathcal{M} \vDash F$", which is language dependend), there are no calculi, for instance PL2 has no complete calculus ("$\mathcal{M} \vDash F \Rightarrow \mathcal{M} \vdash F$"). This is a consequence of Remark 4.9(b): A calculus is in finite steps, i.e. we would have: $\mathcal{M} \vdash F$ is calculated by $\mathcal{M}_0 \vdash F$, $\mathcal{M}_0 \underset{fin}{\subseteq} F$, contradicting the "no compactness" statement.

We have $\{\ulcorner F \urcorner \mid \vDash_{PL2} F\} \notin \Sigma^0_1$