§6 Model theory

6.1 Def: (a) A <u>theory</u> is a set of theorems (in the sense of Def. 4.1 c).

(b) The language $\mathcal{L}(T)$ of a theory T is a PLI1 language with $(\mathcal{C}, \mathcal{F}, \mathcal{P})$ consisting in all constants, functions and predicates used in T. A <u>model</u> of T is a structure $\mathcal{S}$ with $\mathcal{S} \models T$.

(c) The <u>deductive closure</u> $Ded(T) := \{F \mid F\ \mathcal{L}(T)\text{ formula (theorem)}, T \models F\}$ consists in all theorems that can be deduced from T.

6.2 Lemma: Let T be a theory. TFAE:

(i) T is <u>consistent</u> (i.e. $\exists \mathcal{S}$ $\mathcal{L}(T)$-structure: $\mathcal{S} \models T$)

(ii) $Ded(T)$ does not contain a false formula

(iii) $\exists F$ $\mathcal{L}(T)$ formula: $F \notin Ded(T)$

<u>Proof:</u> (i) $\Leftrightarrow$ (ii) ✓

(iii) $\Rightarrow$ (ii): Let $\perp (:= C \wedge \neg C)$ be a false formula.
If $\perp \in Ded(T)$, then $\forall F: F \in Ded(T)$ $(T \models (\perp \to F), T \models \perp \Rightarrow T \models F)$

(i) $\Rightarrow$ (iii): $F \in Ded(T) \Rightarrow \neg F \notin Ded(T)$ □

6.3 Def: (a) Let $\mathcal{L} \subseteq \mathcal{L}'$ be an extension of languages (see Rem. 4.6) and T' be an $\mathcal{L}'$-theory. Then T' is an <u>extension</u> of T $(T \subseteq T')$, if $Ded_{\mathcal{L}}(T) \subseteq Ded_{\mathcal{L}}(T')$.

(b) T' is a <u>conservative extension</u> of T, if $Ded_{\mathcal{L}}(T) = Ded_{\mathcal{L}}(T')$.

(c) T' is a <u>definitional extension</u> of T, if

(i) for any constant $c \in \mathcal{L}(T') \setminus \mathcal{L}(T)$ there is an $\mathcal{L}(T)$ theorem $\exists x F_v^c(x)$ such that $T \models \exists! x\, F_v^c(x)$

(ii) for any function $f \in \mathcal{L}(T') \setminus \mathcal{L}(T)$ there is an $\mathcal{L}(T)$ formula $F^f(v_1, \dots, v_n, u)$ s.t. $FV(F^f) = \{v_1, \dots, v_n, u\}$ and $T \models \forall x_1 \dots x_n \exists! y\, F^f(x_1, \dots, x_n, y)$

(iii) for any predicate $P \in \mathcal{L}(T') \setminus \mathcal{L}(T)$ there is an $\mathcal{L}(T)$ formula F^P, $FV(F^P) = \{v_1, \dots, v_n\}$

(iv) $T' = T \cup \{F^c(c), \forall x_1 \dots x_n F^f(x_1, \dots, x_n, f(x_1, \dots, x_n)), \forall x_1 \dots x_n\, P(x_1, \dots, x_n) \leftrightarrow F^P(x_1, \dots, x_n)\}$

Hence, all new symbols in $\mathcal{L}(T') \setminus \mathcal{L}(T)$ are determined by old ones.

6.4 Example: PLI1 is a conservative extension of PL1.

6.5 Remark: A definitional extension is conservative.

6.6 Def: Let $\mathcal{S}_1 = (S_1, \ldots)$, $\mathcal{S}_2 = (S_2, \ldots)$ be $\mathcal{L}(\mathcal{C}, \mathcal{F}, \mathcal{P})$ structures, $\varphi: S_1 \to S_2$ be a map.

(a) φ is an <u>embedding</u>, if φ is injective and
$\varphi(c^{\mathcal{S}_1}) = c^{\mathcal{S}_2}$, $\varphi(f^{\mathcal{S}_1}(s_1, \ldots s_n)) = f^{\mathcal{S}_2}(\varphi(s_1), \ldots \varphi(s_n))$,
$P^{\mathcal{S}_2} = \{(\varphi(s_1), \ldots \varphi(s_n)) \mid (s_1, \ldots s_n) \in P^{\mathcal{S}_1}\}$.

(b) φ is an <u>elementary embedding</u>, if $\mathcal{S}_1 \models F[\Phi] \Leftrightarrow \mathcal{S}_2 \models F[\varphi \circ \Phi] \ \forall \Phi \forall F$

(c) $\mathcal{S}_1$ is a <u>substructure</u> of $\mathcal{S}_2$ ($\mathcal{S}_1 \subseteq \mathcal{S}_2$), if $S_1 \subseteq S_2$, $id: S_1 \to S_2$ embedding

(d) $\mathcal{S}_1$ is an <u>elementary substructure</u> of $\mathcal{S}_2$ ($\mathcal{S}_1 \prec \mathcal{S}_2$), if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and id is an elementary embedding.

6.7 Def: Let $\mathcal{S}$ be an $\mathcal{L}$-structure.

(a) $D(\mathcal{S}) := \{F \text{ is an atom theorem of } \mathcal{L}_S(\mathcal{S}) \mid \mathcal{S} \models F\}$ <u>diagram</u>

(b) $Th_S(\mathcal{S}) := \{F \text{ is a theorem of } \mathcal{L}_S(\mathcal{S}) \mid \mathcal{S} \models F\}$ <u>elementary diagram</u>

(c) $Th(\mathcal{S}) := \{F \text{ is a theorem of } \mathcal{L}(\mathcal{S}) \mid \mathcal{S} \models F\}$

6.8 Remark: (a) $\mathcal{S}_1 \hookrightarrow \mathcal{S}_2$ embedding $\Leftrightarrow \mathcal{S}_2^{\varphi} \models D(\mathcal{S}_1)$

(b) $\mathcal{S}_1 \hookrightarrow \mathcal{S}_2$ elementary embedding $\Leftrightarrow \mathcal{S}_2^{\varphi} \models Th_{S_1}(\mathcal{S}_1)$

(c) $\mathcal{S}_1 \subseteq \mathcal{S}_2 \Leftrightarrow \mathcal{S}_2 \models D(\mathcal{S}_1)$

(d) $\mathcal{S}_1 \prec \mathcal{S}_2 \Leftrightarrow \mathcal{S}_2 \models Th_{S_1}(\mathcal{S}_1)$ ($\Leftrightarrow \forall F$ thm in $\mathcal{L}_{S_1}$: $\mathcal{S}_1 \models F \Leftrightarrow \mathcal{S}_2 \models F$)

(e) Löwenheim-Skolem: (i) Countable structures are enough:

$\mathcal{L}$ PL1, $\mathcal{S}$ $\mathcal{L}$-str. $\forall M \subseteq S \ \exists \mathfrak{A} \prec \mathcal{S}$ $\mathcal{L}$-str., $|\mathfrak{A}| \leq \aleph_0$
($\Leftrightarrow \mathcal{S} \models Th_A(\mathfrak{A})$, i.e. $\mathfrak{A} \models F \Leftrightarrow \mathcal{S} \models F$)
$\leadsto \mathfrak{A}$ preserves theorems

(ii) $\forall \mathfrak{A}, \aleph_0 \leq |\mathfrak{A}| \ \forall \kappa \geq |\mathfrak{A}| \ \exists \mathfrak{A} \prec \mathcal{S}, |S| = \kappa$

6.9 <u>Def</u>: A theory T is <u>complete</u>, if there is a T model $\mathfrak{A}$ (i.e. $\mathfrak{A} \models T$) such that $Th(\mathfrak{A}) = Ded(T)$.

<u>6.10 Thm</u>: TFAE:

(i) T is complete

(ii) T is consistent and maximal, i.e. $\underbrace{\exists \mathcal{S} : \mathcal{S} \models T}_{T}$ & $\forall F$ $\mathcal{I}(T)$ theorem: $F \in Ded(T) \vee \neg F \in Ded(T)$

(iii) T is consistent and $\forall \mathcal{S} \models T$: $Th_{\mathcal{I}}(\mathcal{S}) = Ded(T)$

<u>Proof</u>: (i) $\Rightarrow$ (ii): T consistent, since $T \subseteq Ded(T) = Th(\mathfrak{A})$, i.e. $\mathfrak{A} \models T$.
If $F \notin Ded(T) = Th(\mathfrak{A})$, then $\mathfrak{A} \not\models F$. Hence $\mathfrak{A} \models \neg F \Rightarrow \neg F \in Th(\mathfrak{A})$

(ii) $\Rightarrow$ (i): T cons. $\Rightarrow \exists \mathfrak{A} \models T$, i.e. $Ded(T) \subseteq Th(\mathfrak{A})$. Need to show „=". $Ded(T)$.
Let $F \notin Ded(T) \overset{(ii)}{\Rightarrow} \neg F \in Ded\,T \subseteq Th(\mathfrak{A}) \Rightarrow \mathfrak{A} \not\models F$, i.e. $F \notin Th(\mathfrak{A})$.

(i) $\Rightarrow$ (iii): T cons. ✓ Now, let $\mathcal{S} \models T$. Then $Ded(T) \subseteq Th_{\mathcal{I}}(\mathcal{S})$.
As in „(ii) $\rightarrow$ (i)", show „=".

(iii) $\Rightarrow$ (i): T cons. $\Rightarrow \exists \mathcal{S} : \mathcal{S} \models T \overset{(iii)}{\Rightarrow} Th(\mathcal{S}) = Ded(T)$. □