§7 Gödel's Incompleteness Theorems

(an embedded, self-contained talk) §7.20

7.0 Aim: Statement of 7.18 : $T \nvdash Cons(T)$ $(PA \subseteq NT \subseteq T)$

7.1 Motivation:
- 19th Century: infinitely small objects in analysis appear
  $\rightsquigarrow$ what is $\mathbb{R}$, what is a set, what is infinity?
  Russell 1903: $R := \{x \mid x \notin x\} \in R \iff R \notin R$
  $\rightsquigarrow$ crisis of mathematics, what are our axioms?
- Hilbert 1900, 2nd problem: Are the Peano axioms (defining $\mathbb{N}$ and number theory) consistent, i.e. free from contradictions?
- Gödel 1931, 2nd Incompleteness Thm: We can't tell!
  No system can prove its own consistency: $T \nvdash Cons(T)$

7.2 Computability:
- What is a computation? How to formalize computability?
- Given a function $f: \mathbb{N} \to \mathbb{N}$, we may understand it by its values:

n	$f(n)$
0	0
1	2
2	4
3	6
$\vdots$	$\vdots$

But how to compute its values? Distinction between function and term.

- PRT $:\overset{1.1}{\iff}$
  - $\underline{S} \in PRT$ $\overset{1.2}{\rightsquigarrow}$ $val(\underline{S}): \mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$
  - $\underline{C}^n_k \in PRT$ $val(\underline{C}^n_k): \mathbb{N}^n \to \mathbb{N}$, $n \mapsto k$
  - $\underline{P}^n_k \in PRT$ $val(\underline{P}^n_k)(x_1, \dots, x_n) = x_k$
  - $\underline{Sub}(\underline{h}, \underline{g}_1 \dots \underline{g}_m) \in PRT$ if $\underline{h}, \underline{g}_i \in PRT$ $val(\underline{Sub}(\dots))$ composition
  - $\underline{Rec}(\underline{g}, \underline{h}) \in PRT$ if $\underline{g}, \underline{h} \in PRT$ $val(\underline{Rec}(\dots))$ Recursion

$f \in PRF :\overset{1.6}{\iff} \exists \underline{f} \in PRT : val(\underline{f}) = f$
primitive recursive functions

$\Downarrow$

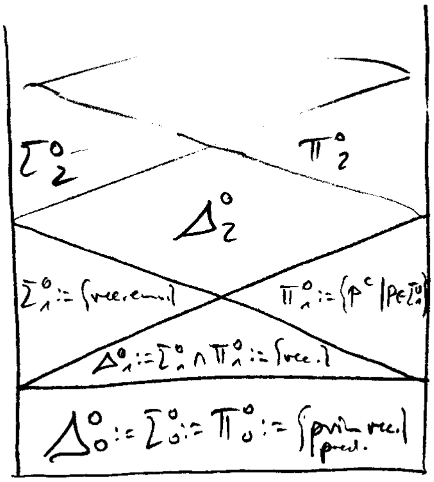
$f \in \mathbb{P} :\overset{2.3}{\iff} \exists \underline{f} \in PRT \cup \{\text{unbounded search for smallest root}\}$
$val(\underline{f}) = f$
partially recursive functions (here: $dom\, f \subsetneq \mathbb{N}^n$ allowed)

formalization of "algorithm"

- $P \subseteq \mathbb{N}^n$:
  - P primitive recursive predicate $:\overset{1.8}{\iff} \chi_P \in PRF$
  - $\Downarrow$ P recursive predicate $:\overset{2.16}{\iff} \chi_P \in \mathbb{P}$, $\operatorname{dom} \chi_P = \mathbb{N}^n$
  - $\Downarrow^{2.19}$ P recursively enumerable predicate $:\overset{2.16}{\iff} P = \operatorname{range}(f_1 - f_n)$, $f_i : \mathbb{N} \to \mathbb{N}$, $f_i \in \mathbb{P}$, $\operatorname{dom} f_i = \mathbb{N}$

- Decidability of membership:

		$n \in P$	$n \notin P$
Δ^0_1	rec.	yes ($\chi_P(n) = 1$)	yes ($\chi_P(n) = 0$)
Σ^0_1	rec. enum.	yes ($\exists x : f(x) = n$)	no! (algorithm never stops!)



$$\Sigma^0_{n+1} \overset{2.23}{:=} \{P \mid \exists R \in \Pi^0_n : \vec{x} \in P \iff \exists y \, R(\vec{x}, y)\}$$

$$= \{\underbrace{\exists y_1 \forall y_2 \exists y_3 \forall y_4}_{n} . \underbrace{R(\vec{x}, \vec{y})}_{\Delta^0_0}\}$$

Ex: General word problem ($\mathcal{L} \subseteq \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ language, $w \in \mathcal{L}$?) $\overset{3.4}{\in} \Sigma^0_1 \setminus \Delta^0_1$
However, if $\mathcal{L}$ "nice" (context sensitive / free): Δ^0_0

7.3 Coding:
- $\ulcorner \underline{S} \urcorner \overset{1.17}{:=} \langle 0, 1 \rangle := p(0)^{0+1} p(1)^{1+1} = 2^1 \cdot 3^2 = 18$, $p : \mathbb{N} \to \mathbb{N}$ prime numbers
  $\ulcorner \underline{C^n_k} \urcorner := \langle 1, n, k \rangle := p(0)^{1+1} p(1)^{n+1} p(2)^{k+1}$
  $\ulcorner \underline{P^n_k} \urcorner := \langle 2, n, k \rangle$ $\qquad \rightsquigarrow \quad \ulcorner \underline{f} \urcorner \in \mathbb{N}$
  $\vdots$
- The procedure of coding and decoding, i.e. $\{n \in \mathbb{N} \mid \text{Is } n \text{ a code?}\} \overset{1.18}{\in} \Delta^0_0$
- Can express complexity questions in terms of membership for natural numbers.

7.4 Formalism of "theory/theorems":

- Language of first order logic$_{(4.1)}$: $\mathcal{L}(\mathcal{C}, \mathcal{F}, \mathcal{P})$ consisting in
  (i) logical symbols $\neg, \wedge, \vee, \rightarrow, \exists, \forall$, variables
  (ii) nonlogical symbols: $\mathcal{C}$ constants, $\mathcal{F}$ functions, $\mathcal{P}$ predicates
- F formula $:\Longleftrightarrow$ $F = P t_1 \ldots t_n$ "atom", where $P \in \mathcal{P}$, t_i (free) variables, constants $c \in \mathcal{C}$ or $f s_1 \ldots s_n$, $f \in \mathcal{F}$, s_i terms
  or $F \in \{\neg A, A \wedge B, A \vee B, A \rightarrow B, \forall x\, A_u(x), \exists x\, A_u(x)\}$
  replace free variable u by bound variable x
- F theorem $:\Longleftrightarrow$ no free variables in F
  Ex: $\forall x: x = x$, where $= \in \mathcal{P}$, $x = x :\Longleftrightarrow =(x, x)$
- T theory $:\Longleftrightarrow$ collection of theorems

7.5 Formalism of deduction:

- "Interpretation"$_{4.2, 4.4}$: $\mathcal{S} = (S, \mathcal{C}^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, \mathcal{P}^{\mathcal{S}})$ is a $\mathcal{L}(\mathcal{C}, \mathcal{F}, \mathcal{P})$ structure, i.e.
  - $S \neq \emptyset$, the support
  - $c^{\mathcal{S}} \in S$, $f^{\mathcal{S}}: S^{\#f} \rightarrow S$, $P^{\mathcal{S}} \subseteq S^{\#P}$ interpretation

  Moreover, Φ: {free variables} $\longrightarrow S$ is needed $\rightsquigarrow$ $t^{\mathcal{S}}[\Phi]$ etc.
- $(P t_1 \ldots t_n)^{\mathcal{S}}[\Phi] := \begin{cases} \text{true} & \text{if } (t_1^{\mathcal{S}}[\Phi], \ldots, t_n^{\mathcal{S}}[\Phi]) \in P^{\mathcal{S}} \\ \text{false} & \text{otherwise} \end{cases}$

  $\rightsquigarrow$ $\mathcal{S} \models F[\Phi]$, if $F^{\mathcal{S}}[\Phi] = \text{true}$, $\mathcal{S} \not\models F[\Phi]$ otherwise, F formula
- $T \models F \;:\overset{4.7}{\Longleftrightarrow}\; \forall \mathcal{S}\, \forall \Phi\, \big[(\mathcal{S} \models G[\Phi] \text{ for all } G \in T) \Rightarrow \mathcal{S} \models F[\Phi] \big]$
- $T \vdash F \;:\overset{5.1}{\Longleftrightarrow}\;$ (i) $F \in M \Rightarrow M \vdash F$
  (ii) $\models_b F$ ("Boolean true") $\Rightarrow M \vdash F$
  (iii) $M \vdash (F_u(t) \rightarrow \exists x\, F_u(x))$ & $M \vdash (\forall x F_u(x) \rightarrow F_u(t))$
  (iv) $M \vdash A$, $M \vdash (A \rightarrow F) \Rightarrow M \vdash F$
  (v) $M \vdash (F \rightarrow A)$ $u \notin FV(M) \cup FV(A) \Rightarrow M \vdash (\exists x F_u(x) \rightarrow A)$
  $M \vdash (A \rightarrow F)$ " $\Rightarrow M \vdash (A \rightarrow \forall x F_u(x))$
- Correctness: $M \vdash F \overset{5.2}{\Longrightarrow} M \models F$
  Completeness: $M \vdash F \overset{5.3}{\Longleftarrow} M \models F$
  $\rightsquigarrow$ $\models F \Longleftrightarrow \vdash F$
  Gödel: Every statement which is true by logical reasons may be deduced already by logical calculus (i.e. $\exists$ proof)
- $\{\ulcorner F \urcorner \mid \models F\} \in \Sigma_1^0 \setminus \Delta_1^0$

7.6 Def: a) $\mathcal{L}_{PA}(\{\underline{0},\underline{1}\},\{\underline{\cdot},\underline{+}\})$ PL I 1 language (i.e. first order & $\{=\}\in P$ with fixed interpretation)

(NF) $\forall x: x+1\neq 0$

(INJ) $\forall x \forall y: (x+1=y+1 \rightarrow x=y)$

(PL0) $\forall x: x+0=x$ — Peano arithmetics

(PL1) $\forall x \forall y: x+(y+1)=(x+y)+1$

(MU0) $\forall x: x\cdot 0=0$

(MU1) $\forall x\forall y: x\cdot(y+1)=xy+x$

(Ind) $(F_u(0)\wedge(\forall x(F_u(x)\rightarrow F_u(x+1)))\rightarrow \forall x\, F_u(x)$

b) $\mathcal{L}_{NT}(\{\underline{0},\underline{1},c_n, n\in\mathbb{N}\},\{PRT\})$

(Kon) $c_0=\underline{0}$, $c_1=\underline{1}$, $c_{n+1}=\underline{S}(c_n)$

(NF0) $\forall x: c_0\neq Sx$

(NF1) $\forall x\forall y: (Sx=Sy \rightarrow x=y)$

(K0) $\forall x_1 \ldots x_n: C^n_k x_1\ldots x_n = c_k$

(PJ) $\forall x_1\ldots x_n: P^n_k x_1 \ldots x_n = x_k$

(Sub) $\forall x_1\ldots x_n \forall y_1 \ldots y_m \forall z\, [h_1 x_1\ldots x_n = y_1, \ldots, h_m x_1 \ldots x_n = y_m,$
$g y_1 \ldots y_m = z \rightarrow \underline{Sub}(g, h_1 \ldots h_m)x_1\ldots x_n = z]$

"Have $\mathbb{N}^2$ in PA"

(Rec) $\forall x_1 \ldots x_n \forall y \forall z \forall u \forall v\, [(y=c_0, g x_1 \ldots x_n = z \rightarrow \underline{Rec}(\underline{g},\underline{h})\, y\, x_1\ldots x_n = z)$
$\wedge (y=\underline{S}u \wedge \underline{Rec}(\underline{g},\underline{h})\, u\, x_1 \ldots x_n = v$
$\rightarrow \underline{Rec}(\underline{g},\underline{h})\, y\, x_1\ldots x_n = h\, x_1 \ldots x_n\, u\, v)]$

(Ind) $F(c_0)\wedge \forall x\, (Fx \rightarrow F(x+1)) \longrightarrow \forall x\, F(x)$

7.7 Lemma: $P(a,b):=(a+b)^2+a+1$ satisfies $P(a,b)=P(a',b')\Rightarrow a=a', b=b'$ in PA

Proof: 1.) $a+b=a'+b'$ since otherwise ("<") $P(a,b)\leq (a+b+1)^2\leq (a'+b')^2 < P(a',b')$

(note: $x\leq y :\Leftrightarrow \exists z: y=x+z$ may be defined in $\mathcal{L}_{PA}$)

2.) with $m:=(a+b)^2$ is $a+1+m=a'+1+m \overset{(INJ)}{\Longrightarrow} a=a' \overset{(INJ)}{\Longrightarrow} b=b'$ □

7.8 Lemma: We may define Gödel coding in $\mathcal{L}_{NT}$

Proof: $\beta(i,b):=\mu x<(b-1): \exists y<b\, \exists z<b\, [b=P(y,z)\wedge (1+(P(i,x)+1)\cdot z)\,|\, y]$

"Gödel β-function" for coding $\langle a_1,\ldots a_n\rangle := \mu x[\beta(0,x)=n \wedge \forall 0<i\leq n: \beta(i,x)=a_i]$

Here: $\beta(i,b)=$ "find x such that $b=(y,z)$ and $y=\langle \ldots, x_i, \ldots\rangle$"

Have: $\forall (a_0,\ldots a_{n-1})\, \exists b\in\mathbb{N}: \beta(i,b)=a_i$ for $i=0,\ldots n-1$ □

7.9 Thm: NT is a definitional extension of PA, i.e. all new symbols in $\mathcal{L}_{NT} \setminus \mathcal{L}_{PA}$ are determined by old ones from $\mathcal{L}_{PA}$ and we have

for all $\mathcal{L}_{PA}$ formulas F: $\quad NT \models F \iff PA \models F$

Proof: Define $\underline{k} := 1 + \ldots + 1$ in $\mathcal{L}_{PA}$ and prove $PA \vdash \exists! x : x = \underline{k}$, i.e. the new constant c_k is determined by $\underline{k}$.
Moreover, define PRT in $\mathcal{L}_{PA}$ using $\underline{k}$ and $+$ (for $\leq$) $\quad \square$

7.10 Remark: $Th(\mathbb{N}) := \{F \ \mathcal{L}_{NT} \text{ formula} \mid FV(F) = \emptyset,\ \mathbb{N} \models F\} \notin \Sigma^0_1$ and there is no system of axioms $A \in \Sigma^0_1$ with $Ded(A) = Th(\mathbb{N})$.
(Theorem by Rosser, preliminary to Gödel) Here $\mathbb{N} := (\omega, 0, 1, +, \cdot)$.

7.11 Def: Let $NT \subseteq T$.

(a) T ω-consistent $:\iff \forall F : \left[(\forall \underline{n} \in \underline{\omega} : T \vdash F_v(\underline{n})) \Rightarrow T \nvdash \exists x\, \neg F_v(x) \right]$

Here $\underline{\omega} := \{ \underbrace{1 + \ldots + 1}_{n} \mid n \in \omega \}$

(b) T ω-complete $:\iff \left[T \vdash F_v(\underline{n})\ \forall \underline{n} \in \underline{\omega} \Rightarrow T \vdash \forall x\, F_v(x) \right]$

7.12 Remark: Def. 7.11 ensures that we have no (strange) models $\mathcal{S} = (S, \ldots)$, $|S| > \omega$. We have $\mathbb{N} \models T \Rightarrow T$ ω-cons.

7.13 Def: (a) A theory T with $NT \subseteq T$ (or rather its coding function) satisfies the deduction principles, if

(1) There are PRPs $\quad Trm(\ulcorner t \urcorner) \iff t \ \mathcal{L}_{NT}$ term
$\quad Fml(\ulcorner F \urcorner) \iff F \ \mathcal{L}_{NT}$ formula

(2) There are PRFs $\quad sub_n : \omega^{n+1} \to \omega$ with $sub_n(\ulcorner E \urcorner, \ulcorner t_1 \urcorner, \ldots, \ulcorner t_n \urcorner) = \ulcorner E_{v_1 \ldots v_n}\, t_1 \ldots t_n \urcorner$
$\quad num : \omega \to \omega$, $n \mapsto \ulcorner \underline{n} \urcorner$

(3) There is a PRP $\quad Bew_T(n, \ulcorner F \urcorner)$ with "n encodes a T proof of F"

We put $\square_T u :\iff \exists x\, Bew_T(x, u)$, i.e. "$u$ may be proven in T"

(b) It satisfies the strong deduction principles, if in addition

(4) $NT \vdash \forall x\, (\square_T x \to \square_T \ulcorner \square_T \dot{x} \urcorner)$
where $\ulcorner F(\dot{x}) \urcorner :\iff sub(\ulcorner F \urcorner, num\, x) = \ulcorner F_v(num\, x) \urcorner$

(5) $NT \vdash \forall x, y\, (\square_T\, imp(x, y) \to (\square_T x \to \square_T y))$
where $imp : \omega^2 \to \omega$, $(\ulcorner A \urcorner, \ulcorner B \urcorner) \mapsto \ulcorner A \to B \urcorner$

We put $Cons(T) := \neg \square_T \bot$ with $\bot := \ulcorner 0 = 1 \urcorner$ false theorem

7.14 Prop.: NT satisfies the strong deduction principles and it is ω-consistent. (by Rem. 7.12)

Proof: (1) For terms: $\ulcorner \underline{0} \urcorner := \langle 5 \rangle$

$\ulcorner \underline{v_i} \urcorner := \langle 7, i \rangle$ free variables

$\ulcorner \underline{x_i} \urcorner := \langle 8, i \rangle$ bound variables

$\ulcorner \underline{f}\, \underline{t_1} \ldots \underline{t_n} \urcorner := \langle 9, \ulcorner \underline{f} \urcorner, \ulcorner t_1 \urcorner, \ldots \ulcorner t_n \urcorner \rangle$

for formulas $\ulcorner t_1 = t_2 \urcorner$, $\ulcorner A \to B \urcorner$ etc.

(2) sub, num $\in$ PRF since detection of $FV(t)$ is PRP

(3) $Bew(n, \ulcorner F \urcorner) :\Leftrightarrow Seq(n) \wedge \forall i < length(n) [(n)_i$ axiom $\vee \exists k < i$ useofquantifiers on $(n)_k, (n)_i \ldots]$

$\wedge\ (n)_{length(n)-1} = \ulcorner F \urcorner$

(4) & (5) Use concatenation (PRF) of proofs. □

7.15 Lemma: If T satisfies the deduction principles, then

(a) $T \vdash F \Rightarrow NT \vdash \Box_T \ulcorner F \urcorner$

(b) T ω-cons., $T \vdash \Box_T \ulcorner F \urcorner \Rightarrow T \vdash F$

If it satisfies the strong ded. princ., then also: (c) $\forall F\ T \vdash (\Box \to F)$ (d) $\forall G: T \vdash (\neg G \leftrightarrow (G \to \bot))$

Proof: (a) $T \vdash F \Rightarrow$ obtain a proof (i.e. a finite sequence of usage of the axioms defining the calculus $\vdash$) and encode it to n. Obtain $Bew_T(n, \ulcorner F \urcorner)$. Hence $NT \vdash \underbrace{\exists x\, Bew_T(x, \ulcorner F \urcorner)}_{\Leftrightarrow\ \Box_T \ulcorner F \urcorner}$

"Σ^0_1 completeness of NT" $\to (\mathbb{N} \models \exists x\, Bew_T(x, \ulcorner F \urcorner)$ & $NT \vdash G$ $\forall G \to \mathbb{N} \models G)$

(b) Assume $T \nvdash F$, i.e. $\forall n \in \omega\ T \vdash \neg Bew_{T,n}(\underline{n}, \ulcorner F \urcorner)$

$\overset{T\ \omega\text{-cons.}}{\Longrightarrow} T \nvdash \underbrace{\exists x\, Bew_T(x, \ulcorner F \urcorner)}_{\Box_T \ulcorner F \urcorner}$. But $T \vdash \Box_T \ulcorner F \urcorner$. ↯ □

7.16 Arithmetic Fixed Point Thm: Let F be an $\mathcal{L}_{NT}$ formula, $FV(F) = \{v\}$. Then there is a $\mathcal{L}_{NT}$ theorem A with $NT \vdash (A \leftrightarrow F_v(\ulcorner A \urcorner))$.

Proof: Put $H := F_v(sub(w, num(w)))$ and $A := H_w(\ulcorner H \urcorner)$.

Then $NT \vdash [A \leftrightarrow H_w(\ulcorner \underline{H} \urcorner) \leftrightarrow F_v(sub(\ulcorner H \urcorner, num \ulcorner H \urcorner))$

$\leftrightarrow F_v(\ulcorner H_w(\ulcorner H \urcorner) \urcorner) \leftrightarrow F_v(\ulcorner A \urcorner)]$. □

7.17 First Incompleteness Thm (Gödel 1931, "Thm VI"):

[Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatshefte für Mathematik, 1931] & Rosser 1936

Let T be a consistent extension of NT with $\mathbb{N} \models T$ satisfying the deduction principles. There is a Π_1^0 theorem G_T with $NT \vdash (G_T \Leftrightarrow \neg \Box_T \ulcorner G_T \urcorner)$ and $T \nvdash G_T$ and $\mathbb{N} \models G_T$.
If T is ω-consistent, then also $T \nvdash \neg G_T$.
Hence, in this case: $T \nvdash G_T$ and $T \nvdash \neg G_T$.

Proof: Put $G_T := A$ from Thm 7.16 with $F := \neg \Box_T v$.
Assume $T \vdash G_T$. Then $NT \vdash \Box_T \ulcorner G_T \urcorner$ by Lemma 7.15(a).
Hence $NT \vdash \neg G_T$ by assumption $(NT \vdash (G_T \Leftrightarrow \neg \Box_T \ulcorner G_T \urcorner))$.
$\xRightarrow{Ded(NT) \subseteq Ded(T)} T \vdash \neg G_T$ ↯ (T consistent)
Now assume that T is ω-consistent and $T \vdash \neg G_T$. Then
$T \vdash \Box_T \ulcorner G_T \urcorner \xRightarrow{L7.15(b)} T \vdash G_T$ ↯ (T consistent).
As for $\mathbb{N} \models G_T$: $\mathbb{N} \models \neg Bew_T(n, \ulcorner G_T \urcorner)$ $\forall n \in \omega$ since $T \nvdash G_T$.
Thus, $\mathbb{N} \models \forall x \neg Bew_T(n, \ulcorner G_T \urcorner)$ ($\mathbb{N}$ is ω-complete).
Hence, $\mathbb{N} \models \neg \Box_T \ulcorner G_T \urcorner \xRightarrow{NT \vdash (G_T \Leftrightarrow \neg \Box_T \ulcorner G_T \urcorner)} \mathbb{N} \models G_T$. $\Box$

7.18 Second Incompleteness Thm (Gödel 1931, "Thm XI"):

Let T be a consistent extension of NT with $\mathbb{N} \models T$ satisfying the strong deduction principles. Then $T \nvdash Cons(T)$.

Proof: We have $T \nvdash G_T$ by Thm. 7.17 (1st Incompl. Thm).
We have to show $NT \vdash (G_T \Leftrightarrow Cons(T))$.
"$\Rightarrow$": By lemma 7.15c, we have $T \vdash (\bot \to G_T) \xRightarrow{7.15(a)} NT \vdash \Box_T(\ulcorner \bot \to G_T \urcorner)$
$\xRightarrow{7.13(5)} NT \vdash (\Box_T \ulcorner \bot \urcorner \to \Box_T \ulcorner G_T \urcorner) \xRightarrow{\text{Contrapos.}} NT \vdash (\neg \Box_T \ulcorner G_T \urcorner \to \neg \Box_T \ulcorner \bot \urcorner)$
$\xRightarrow{7.17} NT \vdash (G_T \to Cons(T))$, since $Cons(T) = \neg \Box_T \ulcorner \bot \urcorner$.

„$\Leftarrow$" We will show $NT \vdash (\neg G_T \to \neg Cons(T))$ (Contrapos.)

By 7.15(d), we have $T \vdash (G_T \to (\neg G_T \to \bot))$

$\overset{7.15(a)}{\Longrightarrow}$ $NT \vdash \Box_T \ulcorner G_T \to (\neg G_T \to \bot)\urcorner$

$\overset{7.13(5)}{\Longrightarrow}$ $NT \vdash \Box_T\ulcorner G_T\urcorner \to (\Box_T \ulcorner \neg G_T \urcorner \to \Box_T\ulcorner\bot\urcorner)$ $[NT \vdash \neg A \vee \neg B \vee C]$

Moreover, we have $NT \vdash (\Box_T\ulcorner G_T\urcorner \to \neg G_T)$ by 7.17.

$\overset{7.15(a)}{\Longrightarrow}$ $NT \vdash \Box_T \ulcorner \Box_T\ulcorner G_T\urcorner \to \neg G_T\urcorner$

Looks already like a contradiction $\overset{7.13(5)}{\Longrightarrow}$ $NT \vdash \Box_T \ulcorner \Box_T\ulcorner G_T\urcorner\urcorner \to \Box_T\ulcorner\neg G_T\urcorner$ $[NT \vdash \neg D \vee B]$

Therefore, combining the two statements, we obtain

$NT \vdash \Box_T\ulcorner G_T\urcorner \to (\Box_T\ulcorner\Box_T\ulcorner G_T\urcorner\urcorner \to \Box_T\ulcorner\bot\urcorner)$ $[NT \vdash \neg A \vee \neg D \vee C]$

Now, $NT \vdash \Box_T\ulcorner G_T\urcorner \to \Box_T\ulcorner\Box_T\ulcorner G_T\urcorner\urcorner$ by 7.13(4).

Summarizing: $NT \vdash \underbrace{\Box_T\ulcorner G_T\urcorner}_{\neg G_T} \to \underbrace{\Box_T\ulcorner\bot\urcorner}_{\neg Cons(T)}$

$\Box$

7.19 Remarks: We may show (under the assumptions of 7.18):

(a) $NT \vdash (Cons(T) \Leftrightarrow \neg\Box_T\ulcorner Cons(T)\urcorner)$

(b) $NT \vdash (Cons(T) \Leftrightarrow Cons(T + \neg Cons\,T))$

(c) Löb: $T \vdash (\Box_T\ulcorner F\urcorner \to F) \Leftrightarrow T \vdash F$

(d) $\forall F: T \vdash \neg F \Rightarrow T \nvdash (\Box_T\ulcorner F\urcorner \to F)$

7.20 Remark: The main ingredients of Gödel's theorems are:

- Using coding, we may express meta phenomena within $\mathbb{N}$.
- We may even apply meta structures on itself:
  $T \vdash F \rightsquigarrow T \vdash \Box_T\ulcorner F\urcorner$
- This way, we may use some diagonal arguments to produce fixed points $A \Leftrightarrow F_V(\ulcorner A\urcorner) \rightsquigarrow G_T \Leftrightarrow \neg\Box_T\ulcorner G_T\urcorner$