II Set Theory & Forcing

Outline: The goal is to understand ZFC as the axioms of set theory. Moreover, we want to prove:

- Cons ZF $\Rightarrow$ Cons ZF + $\bar{V} = L$, for some "constructive universe L"
- Cons ZF $\Rightarrow$ Cons ZFC (via $ZF + (\bar{V}=L) \vdash AC$)
- $ZF + (\bar{V}=L) \vdash CH$, i.e. Cons ZF(C) $\Rightarrow$ Cons(ZFC + CH)
- Cons ZFC $\Rightarrow$ cons(ZFC + $\neg$CH)

ZFC:

ZF:

- $\boxed{(\text{Ext})}$ $\forall x \forall y\ [\forall z\,(z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$
- $\boxed{(\exists \emptyset)}$ $\emptyset \in \bar{V}$
- $\boxed{(\text{Pair})}$ $\forall x, y\ \ \{x, y\} \in \bar{V}$
- $\boxed{(\text{Union})}$ $\forall x\ \ \bigcup x \in \bar{V}$
- $\boxed{(\text{Sep})}$ $A \cap x \in \bar{V}$ for all classes A
- $\boxed{(\text{Repl})}$ F function $\Rightarrow F''x \in \bar{V}$
- $\boxed{(\text{Power})}$ $\forall x\ \ \mathcal{P}(x) \in \bar{V}$
- $\boxed{(\text{Found})}$ $A \neq \emptyset \Rightarrow \exists x$ $\in$-min. in A
- $\boxed{(\text{Inf})}$ $\omega \in \bar{V}$

$\boxed{(\text{AC})}$ Axiom of choice

28 May	ZF, ordinal numbers
4 June	AC, cardinal numbers
11 June	ultrafilters, inner models, cons ZFC $\Rightarrow$ cons CH
18 June	(no lecture)
25 June	$\boxed{\text{Forcing lecture}}$ + details
2 July	(no lecture)
9 July	lecture by Philipp Lücke
16 July	Calkin?

§ 8 The axioms ZF without (Inf)

8.1 Def: (a) $\bar V$ is the universe satisfying the axioms of ZF. Elements $x \in \bar V$ are called sets.

(b) A class is given by $A = \{x \in \bar V \mid \varphi(x, y_1, \dots, y_m)\}$ where φ is a formula defined by

(i) $\varphi := (a \in b)$ or $\varphi := (a = b)$

(ii) $\varphi := (\neg \varphi_1)$, $\varphi := (\varphi_1 \wedge \varphi_2)$, $\varphi := (\varphi_1 \vee \varphi_2)$, $\varphi := (\varphi_1 \to \varphi_2)$

(iii) $\varphi := (\forall x \in \bar V\, \varphi_1(x))$, $\varphi := (\exists x \in \bar V\, \varphi_1(x))$

8.2 Remark: (a) $x \in \bar V \Rightarrow x$ is a class $[x = \{y \mid y \in x\}]$

(b) $\bar V = \{x \mid x = x\}$ is a class but no set. (see 8.7)

(c) $A := \{x \mid x \notin x\}$ is a class but no set. (Russell)

8.3 Def: We may define $A \subseteq B :\iff \forall x (x \in A \Rightarrow x \in B)$ for classes A, B. Likewise $A \cap B$, $A \cup B$, $\neg A$, $A \setminus B$. Also

$\bigcap A := \{x \mid \forall y [y \in A \Rightarrow x \in y]\}$ "inner intersection/union"

$\bigcup A := \{x \mid \exists y [y \in A \wedge x \in y]\}$

Also $A \in B :\iff \exists x [x \in B \wedge x = A]$ (i.e. A is a set)

And $\{x_1, \dots x_n\} := \{x \mid x = x_1 \vee \dots \vee x = x_n\}$ for $x_1, \dots x_n \in \bar V$

(Ex.: $A = \{\{0\}, \{1,2\}, \{1,3\}\}$, $\bigcup A = \{0,1,2,3\}$)

8.4 Def: $\boxed{(\mathrm{Ext})}$ $\forall x \forall y \, [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$ (same elements ⇒ same set)
$(\forall x \in \bar V)$

$\boxed{(\exists \emptyset)}$ $\emptyset := \{x \mid x \neq x\} \in \bar V$ (base case – the only one!)

$\boxed{(\mathrm{Pair})}$ $\forall x, y : \{x, y\} \in \bar V$ (i.e. $\forall x \forall y \exists z : [x \in z \wedge y \in z \wedge \forall w \in z (w = x \vee w = y)]$)

$\boxed{(\mathrm{Union})}$ $\forall x \ \bigcup x \in \bar V$

8.5 Remark: (a) $0 := \emptyset$, $1 := \{\emptyset\}$, $2 := \{\emptyset, \{\emptyset\}\}$, $n+1 := \{n\} \cup n$ all in $\bar V$
$[x = y = \emptyset, (\mathrm{Pair}) \Rightarrow \{\emptyset\} \in \bar V]$

(b) $\bigcap \bar V = \emptyset$, $\bigcup \bar V = \bar V$, $x \cup y \in \bar V$, $\{x_1, \dots x_n\} \in \bar V$
"$\bigcup \{x, y\}$"

(c) ordered pairs $\langle x, y \rangle := \{\{x\}, \{x, y\}\}$, $\langle x_1, \dots x_n \rangle := \langle x_1, \langle x_2, \dots x_n \rangle \rangle$ are in $\bar V$ and $\langle x, y \rangle = \langle x', y' \rangle \iff x = x', y = y'$

May hence define $A \times B := \{\langle x, y \rangle \mid x \in A \wedge y \in B\}$, A^n etc.

8.6 Def: $\boxed{(Sep)}$ $\forall x: A \cap x \in \bar{V}$ whenever A is a class
(no single axiom but a principle of axioms $((Sep)_A)_{A \text{ class}}$)

8.7 Remark: (a) $A = \{z \mid \varphi(z, \vec{y})\}$, $A \cap x = \{z \mid \varphi(z,\vec{y}) \wedge z \in x\} \in \bar{V}$
(b) $\exists x \in \bar{V}: A \subseteq x \Rightarrow A \in \bar{V}$
(c) $\bigcap A \in \bar{V}$ $[x \in A \neq \emptyset$. Then $\bigcap A \subseteq x]$
(d) $\bar{V} \notin \bar{V}$ since otherwise $A \in \bar{V}$ for all classes $\lightning$ (Russell)
(e) $\neg x \notin \bar{V}$ $[\bar{V} = x \cup \neg x]$

8.8 Def: (a) $R \subseteq \bar{V}^2$ is called a relation, write xRy or $R\langle x,y\rangle$.
(b) $R''A := \{y \mid \exists x \in A: yRx\}$
(c) $F \subseteq \bar{V}^2$ is a function, if $\forall x,y,z: [yFx \wedge zFx] \Rightarrow z = y$
We also write $F(x)$ for y with yFx (uniquely determined)
We also write $F: A \to B$, if $A = \{x \mid \exists y: yFx\}$ and $\{y \mid \exists x\ yFx\} \subseteq B$. Likewise injective, surjective, ...

8.9 Def: $\boxed{(Repl)}$ F function. $F''x \in \bar{V}$ for all $x \in \bar{V}$

8.10 Remark: (a) (Repl) $\rightarrow$ (Sep)
(b) $u \times v \in \bar{V}$

8.11 Def: $\boxed{(Power)}$ $\forall x$ $\mathbb{P}(x) := \{z \mid z \subseteq x\} \in \bar{V}$
$\boxed{(Min)}$ If $A \neq \emptyset$, then there is $x \in \bar{V}$ which is $\in$-minimal in A, i.e. $\forall y \in x: y \notin A$ and $x \in A$.
(avoids $x_1 \in x_2 \in \dots \in x_n \in x_1$ and $x \in x$)
More general, x R-minimal in A, if $x \in A$ and $\forall y: yRx \Rightarrow y \notin A$.

@ 8.10: (c) $R \subseteq \bar{V}^2$ set-like $:\Longleftrightarrow \forall x\ R''x \in \bar{V}$
$\Longleftrightarrow \forall x\ R''\{x\} = \{y \mid yRx\} \in V$