§9 Ordinal numbers and (Inf)

9.1 Def.: Let $R \subseteq \bar{V}^2$ be a relation and U a class.

(a) R is founded $:\iff$ any nonempty class A has an R-minimal element

(b) R is strongly founded $:\iff$ R is founded and set-like

(c) $\langle U, R\rangle$ is a (strong) well-ordering, if R is (strongly) founded and $\langle U, R\rangle$ is a linear ordering, i.e. for all $x, y, z \in U$: $\neg(xRx)$, $xRyRz \Rightarrow xRz$, $xRy \vee yRx \vee x=y$

9.2 Remark: (a) There is a Recursion Thm: R str. founded, $R \subseteq A^2$, $G: A \times V \to B$ given $\Rightarrow \exists! F: A \to B$ with $\forall x \in A: \quad F(x) = G(x, F \restriction R''\{x\})$

where $F \restriction C := \{\langle y, x\rangle \mid yFx \wedge x \in C\}$ elements preceding x

(b) There is Mostowski's Isomorphism Thm:

Or, more specific: Let $\langle U, R\rangle$ be a str. well-ordering.

$\boxed{\text{Let } \langle U, R\rangle \text{ be strongly founded and extensional } (\text{i.e. } \forall x, y \in U: R''\{x\} = R''\{y\} \Rightarrow x = y).}$ Then there is a unique transitive class V (i.e. $\forall y \in V\ \forall x \in y: x \in V$) and a unique $F: \langle U, R\rangle \longrightarrow \langle V, \in \restriction V\rangle$ (the "Mostowski collapse") such that $F: U \to V$ bijective and $xRy \iff Fx \in Fy$.
Here, F is given by $F(x) = F''(R''\{x\}) = \{F(y) \mid yRx\}$, $x \in U$ (by (a))
Hence, $\langle V, \in\rangle$ is all we need to study.
("F collapses the holes in non-transitive classes")

9.3 Def.: (a) Let $\langle U, R\rangle$ be a strong well-ordering and $F: \langle U, R\rangle \to \langle V, \in \restriction V\rangle$ be its collapse. Put $|\langle U, R\rangle| := V$.
$On := \{|\langle U, R\rangle| \mid \langle U, R\rangle \text{ str. well-ordering}\} \subseteq \bar{V}$ ordinal numbers

(b) $\omega := \{|\langle U, R\rangle| \mid \langle U, R\rangle \text{ finite linear ordering}\} \subseteq On$
i.e. $\langle U, R\rangle$ & $\langle U, R^{-1}\rangle$ well-orderings

(why "finite"? For $A \cap U \neq \emptyset$, there is a unique (by $xRy \vee yRx \vee x=y$) R-minimal element $x \in A$ and likewise a unique R^{-1}-minimal one, $y \in A$.
Now R^{-1}-minimal $=$ R-maximal, i.e. $A \cap U$ is "finite".)

(c) We put $\alpha < \beta :\iff \alpha \in \beta$ for $\alpha, \beta \in On$.

9.4 Remark: (a) On is a class but no set. It's complexity is Δ^0_0.

$\Big[On \notin \bar{V}$: $\langle On, \underset{\uparrow 9.3c}{<} \rangle$ is a strong well-ordering, hence $On \in \bar{V}$ would imply $On \in On$ ↯ $\Big]$

(On, ω transitive, no collapsing) $\longrightarrow$ $\overset{\shortparallel}{|\langle On, < \rangle|}$

(b) $\omega \notin \bar{V} \implies \omega = On$ $\quad\Big[\omega \subsetneqq On \implies \omega \in On \implies \omega \in \bar{V}\Big]$

9.5 Def.: $\boxed{(Inf)}$ $\omega \in \bar{V}$ (guarantees that $\omega \subsetneqq On$, enough ordinals)

9.6 Remark: (a) Putting $S\alpha := \alpha \cup \{\alpha\}$ for $\alpha \in On$, we have
$0 \in On$, $S\alpha \in On$ for $\alpha \in On$, $\alpha < \beta \iff S\alpha \leq \beta$ etc.
Also $n \in \omega$ for n from Rem. 8.5a.

(b) $\langle U, R \rangle$ finite lin. ordering $\iff \exists n \in \omega\, \exists f : U \to n$ bij.
Define $\overline{\overline{\langle U, R \rangle}} := n$ its cardinality. (n unique)
Have $\overline{\overline{u \cup v}} = \bar{\bar{u}} + \bar{\bar{v}}$, $\overline{\overline{u \times v}} = \bar{\bar{u}} \cdot \bar{\bar{v}}$, $\overline{\overline{P(u)}} = 2^{\bar{\bar{u}}}$ for u finite (i.e. $\langle u, R \rangle$ finite lin. ordering)

(c) Put $\bar{V}_0 := \emptyset$, $\bar{V}_{n+1} := P(\bar{V}_n)$, $\bar{V}_\omega := \bigcup_{n \in \omega} \bar{V}_n$.
Then $\bar{V}_n \neq n$ in general ($3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \not\ni \{\{\emptyset\}\}$)
And $\bar{V}_n, \bar{V}_\omega \in \bar{V}$, $\bar{V}_n \cap On = n$, $m < n \implies \bar{V}_m \in \bar{V}_n$ etc.
Also $\bar{V}_\omega \neq \bar{V}$ by (Inf). $[\omega \notin \bar{V}_\omega]$
We have $(ZF - (Inf) + \neg(Inf)) \vdash (\bar{V} = \bar{V}_\omega)$

(d) We say that $\alpha \in On$ is a <u>limit ordinal</u> ($\lim(\alpha)$), if
$\forall \beta < \alpha\, \exists \gamma < \alpha : \beta < \gamma$ (hence $\alpha \neq S\alpha'$, no successor ordinal)
The smallest limit ordinal is ω. $[n \in \omega \implies Sn \in \omega \implies \lim(\omega)]$
& $n \in \omega \implies n = 0 \vee n = Sn', n' \in n$

(e) May define $\sup A := \bigcup A$ for $A \subseteq On$. Then $\lim \alpha \implies \sup \alpha = \alpha$

(f) Induction principle for On: Let $A \subseteq On$ be such that
$0 \in A$, $\forall \alpha : \alpha \in A \implies S\alpha \in A$, $\forall \alpha ; \lim(\alpha) \wedge \alpha \subseteq A \implies \alpha \in A$.
Then $A = On$.
Note: omitting "$\forall \alpha \lim(\alpha) \dots$" and assuming "$A \subseteq \omega$", we obtain $A = \omega$.

(g) $\bar{V}_\alpha := \bigcup_{\beta < \alpha} \bar{V}_\beta$ for $\lim \alpha$ extends (c) for limit ordinals.

We still have $\alpha < \beta \implies \bar{V}_\alpha \in \bar{V}_\beta$, $\bar{V}_\alpha \cap On = \alpha$ etc.
$\bar{V}_\infty := \bigcup_{\alpha \in On} \bar{V}_\alpha$ satisfies $\bar{V}_\infty = \bar{V}$. $\Big[x \in \bar{V} \setminus \bar{V}_\infty$ $\in$-minimal $\implies x \subseteq \bar{V}_\infty \leadsto x \in \bar{V}_\alpha \in \bar{V}_\infty$ ↯ $\Big]$