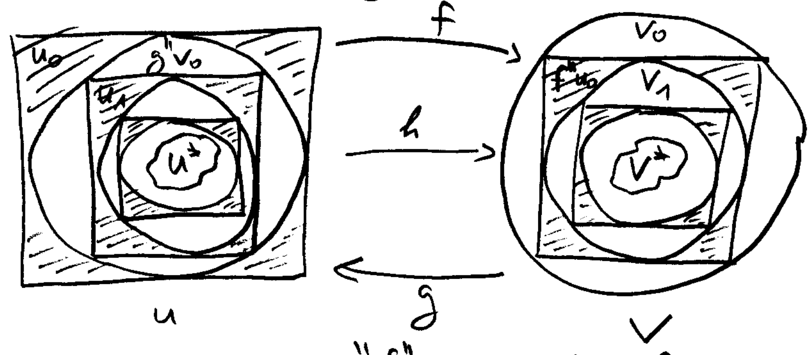
§10 Cardinal numbers and (AC)

10.1 Def.: (a) $u \sim v :\iff \exists f: u \to v$ bijective, $u, v \in \bar{V}$

(b) $u \preceq v :\iff \exists f: u \to v$ injective

10.2 Thm. (Schröder-Bernstein): $u \preceq v \wedge v \preceq u \iff u \sim v$

Proof: "$\Rightarrow$" Let $f: u \to v$, $g: v \to u$ be injective.



$h := \begin{cases} f|_{\equiv \,\&\, u^*} \\ g^{-1}|_{\text{otherwise}} \end{cases}$

$u_0 := u$, $u_{n+1} := g''f''u_n$, $u^* := \bigcap_{n\in\omega} u_n$

$v_0 := v$, $v_{n+1} := f''g''v_n$, $v^* := \bigcap_{n\in\omega} v_n$

Then $u_n \supseteq g''v_n \supseteq u_{n+1}$, $v_n \supseteq f''u_n \supseteq v_{n+1}$, $\bigcap f''u_n = v^*$, $f''u^* = v^*$

$$h(x) := \begin{cases} f(x) & x \in u^* \text{ or } x \in u_n \setminus g''v_n \text{ for some } n \in \omega \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

□

10.3 Lemma: $u \in \bar{V}$. Then $u \not\sim P(u)$.

Proof: A: $\exists f: u \twoheadrightarrow P(u)$. Put $a := \{z \in u \mid z \notin f(z)\} \subseteq u \overset{(Sep)}{\Longrightarrow} a \in P(u)$.

Let $y \in u$ with $a = f(y)$. Then $y \in a \iff y \notin a$ ↯ □

10.4 Def.: f choice function for $u \in \bar{V} :\iff f: u \to \bigcup u$ s.t. $\forall x \in u: f(x) \in x$

10.5 Prop.: TFAE:

(i) $\boxed{(AC)}$ $\forall x$ with $(\forall t \in x: t \neq \emptyset)$ $\exists$ choice function

(ii) $\langle u_t \mid t \in v \rangle \in \bar{V}$, $\forall t \in v: u_t \neq \emptyset \implies \prod_{t \in v} u_t \neq \emptyset$, $\prod_{t\in v} u_t := \{f: v \to \bigcup_{t \in v} u_t,\ f(t) \in u_t\}$

(iii) $\forall u\, \exists r: \langle u, r \rangle$ well-ordering

(iv) $\forall u, v: u \preceq v$ or $v \preceq u$

(v) [Zorn] $\forall u$ chain closed [i.e. if $k \subseteq u$ is a chain $(\forall x, y \in k: x \subseteq y$ or $y \subseteq x)$ then $\bigcup k \in u$] we find a maximal element $m \in u$ (i.e. $\nexists m' \in u: m \subsetneq m'$).

Proof: (i) => (ii): $X := \{u_t \mid t \in v\}$ has a choice function f, put $\tilde{f}(t) := f(u_t)$.

(ii) => (iii): Use $\prod_{a \in \mathcal{P}(u)\setminus\{u\}} u\setminus a \neq \emptyset$ to obtain f with $f(a) \in u\setminus a$.

Use this to find some $G : On \to u$ with $\operatorname{dom} G \sim u$.

(iii) => (iv): $u, v \in \tilde{V} \overset{(iii)}{\Longrightarrow} \exists \langle u, r\rangle, \langle v, s\rangle$ wellorderings.

By the collapse (9.26), we have $|\langle u,r\rangle| \leq |\langle v,s\rangle|$ or $|\langle u,r\rangle| \geq |\langle v,s\rangle|$.

(iv) => (iii): $u^* := \{\langle v, r\rangle \text{ well-ordering} \mid v \subseteq u\} \subseteq \mathcal{P}(u) \times \mathcal{P}(u^2) \Rightarrow u^* \in \tilde{V}$.

For $\alpha := \sup\{\beta + 1 \mid \beta \in F''u^*\}$, $F : u^* \to On$, $\langle v, r\rangle \mapsto |\langle v, r\rangle|$

have $\alpha \not\preceq u$ since α is larger than the largest type $|\langle v,r\rangle|$ on u.

By (IV): $\exists f : u \to \alpha$ inj., put $x r y :\Leftrightarrow f(x) r f(y)$.

(iii) => (i): For x with $t \neq \emptyset$ $\forall t \in x$, find $\langle \bigcup x, r\rangle$ well-ordering.

Put $f : x \to \bigcup x$ choice function.
$x \mapsto r\text{-min. el. of } x$

(i) => (v): u chain closed. Find $\langle u, r\rangle$ well-ordering.

Define $F : On \to a$ partially
$0 \mapsto r\text{-min. el. of } a$
$\alpha+1 \mapsto r\text{-min. el. } x \in a$ with $F(\alpha) \subsetneq x$ if ex. (undef. otherwise)
$\lim \alpha \mapsto \bigcup F''\alpha$ if $F''\alpha$ chain (undef. otherwise)

Then $\operatorname{dom} F = \alpha + 1 \Rightarrow F(\alpha)$ maximal.

(v) => (i): $x' := \{f \mid \operatorname{dom} f \subseteq x, \forall x \in \operatorname{dom} f : f(x) \in x\}$ chain closed

=> $\exists f$ maximal in x', hence $\operatorname{dom} f = x$ and f choice function. □

10.6 Remark: $u \neq \emptyset$. $u \preceq v :\Leftrightarrow \exists f : u \to v$ inj. $\overset{ZF}{\Longrightarrow}\ \underset{ZFC}{\Longleftarrow}$ $\exists g : v \to u$ surj.

Proof: "=>": $g(y) := \begin{cases} f^{-1}(y) & y \in f''u \\ x_0 & \text{otherwise} \end{cases}$, some $x_0 \in v$.

"<=": $\prod_{x \in u} g^{-1}{}''\{x\} \neq \emptyset$ by (AC).

10.7 Def: $u \in \tilde{V}$. $\bar{\bar{u}} := \min\{|\langle u,r\rangle| \mid \langle u, r\rangle \text{ well-ordering}\}$, $Card := \{\alpha \mid \bar{\bar{\alpha}} = \alpha\}$

in particular $u \sim \bar{\bar{u}}$

10.8 Lemma: (a) $u \sim v \Leftrightarrow \bar{\bar{u}} = \bar{\bar{v}}$

(b) $\forall \alpha$: $\exists u : \alpha = \bar{\bar{u}} \Leftrightarrow \alpha \in Card$

(c) $u \preceq v \Leftrightarrow \bar{\bar{u}} \leq \bar{\bar{v}}$

Proof: (a) "=>" $\bar{\bar{u}} \sim u \sim v \sim \bar{\bar{v}}$ "<=" $u \sim \bar{\bar{u}} = \bar{\bar{v}} \sim v$

(b) "=>" $\alpha = \bar{\bar{u}} \Rightarrow \alpha \sim u \overset{(a)}{\Rightarrow} \bar{\bar{\alpha}} = \bar{\bar{u}} = \alpha$. "<=" $u := \alpha$

(c) "<=" $u \sim \bar{\bar{u}} \subseteq \bar{\bar{v}} \sim v \Rightarrow \exists f : u \to v$ inj. "=>" $\bar{\bar{u}} \not\leq \bar{\bar{v}} \Rightarrow \bar{\bar{v}} \subsetneq \bar{\bar{u}}$

$\overset{\text{"<="}}{\Longrightarrow} v \preceq u$, $\not\!$ 10.2.

□

10.9 Remark: (a) Card $\subsetneq$ On $\left[\omega+1 \sim \omega \text{ via } f: \omega+1 \to \omega,\ a \mapsto \begin{cases} a+1 & a<\omega \\ 0 & a=\omega \end{cases} \right]$

10.10 Def: (a) $\langle \aleph_\alpha, \alpha \in On \rangle$ enumeration of Card$\setminus\omega$, $\aleph_0 = \omega$

(b) $\alpha^+ := \min \text{Card} \setminus (\alpha+1)$ $\left(\aleph_{\alpha+1} = (\aleph_\alpha)^+, \aleph_\lambda = \sup_{\alpha<\lambda} \aleph_\alpha \text{ for } \lim \lambda \right)$

10.11 Remark: For $\alpha+\beta := \overline{\overline{(\alpha\times\{0\}) \cup (\beta\times\{1\})}}$, $\alpha\cdot\beta := \overline{\overline{\alpha\times\beta}}$, $\alpha,\beta \in$ Card:

(a) $\overline{\overline{u\times v}} = \overline{\overline{u}} \cdot \overline{\overline{v}}$

(b) $\alpha+0=\alpha$, $\alpha+\beta=\beta+\alpha$, $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$

(c) $\alpha\beta=\beta\alpha$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, $\alpha 1 = \alpha$

(d) $\alpha\cdot 0 = 0$, $\alpha\cdot 2 = \alpha+\alpha$

(e) $\beta \geq \omega$, $\alpha > 0$. , $\alpha\beta = \alpha+\beta = \max(\alpha,\beta)$

in particular $\alpha\alpha = \alpha$ for $\alpha \geq \omega$

(f) $\alpha^\beta := \overline{\overline{\{f:\beta\to\alpha\}}}$, $\alpha^{(\beta+\gamma)} = \alpha^\beta \alpha^\gamma$, $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$, $2^\alpha = \overline{\overline{\mathcal{P}(\alpha)}}$

10.12 Def: (a) x cofinal in α $:\iff x \subseteq \alpha$ and $\forall \gamma<\alpha\ \exists \delta \in x: \delta \geq \gamma$

"can exhaust α by β" $f:\beta\to\alpha$ cofinal $:\iff$ ran $f \subseteq \alpha$ cofinal $(\forall\gamma<\alpha\ \exists\delta\in x: f(\delta)\geq\gamma)$

(b) Assume $\lim(\alpha)$. $cf(\alpha) := \min\{\beta \mid \exists f:\beta\to\alpha \text{ cofinal}\}$

α regular $:\iff cf(\alpha)=\alpha$, α singular otherwise.

10.13 Remark: (a) $\aleph_\omega = \sup_{n\in\omega} \aleph_n$. Hence $cf(\aleph_\omega) \underset{\uparrow}{=} \omega \neq \aleph_\omega$, $cf(\aleph_{\aleph_1}) = \aleph_1$

(have $f:\omega\to\aleph_\omega,\ n\mapsto\aleph_n$)

(b) $\lim\lambda$. Then $\lim cf(\lambda)$, $cf(cf(\lambda)) = cf(\lambda)$, $cf(\lambda) \leq \overline{\overline{\lambda}} \leq \lambda$, $cf(\lambda) \in$ Card.

(c) ω regular, $\kappa \in \text{Card}\setminus\omega \Rightarrow \kappa^+$ regular

[Assume $\alpha := cf(\kappa^+) < \kappa^+ \overset{\alpha\in\text{Card}}{\Rightarrow} \alpha \leq \kappa$. May then construct $h:\kappa\times\alpha\to\kappa^+$ surj., but $\overline{\overline{\kappa\times\alpha}} = \max(\overline{\overline{\kappa}},\overline{\overline{\alpha}}) \leq \kappa$ & $\kappa^+ \leq \kappa\times\alpha$ $\Rightarrow \kappa^+ \leq \kappa$ ↯

(d) $\kappa \in \text{Card}\setminus\omega \Rightarrow 2^\kappa > \kappa$, $\kappa^{cf(\kappa)} > \kappa$, $cf(2^\kappa) > \kappa$.

10.14 Def: GCH : $\forall \kappa \in \text{Card}\setminus\omega: 2^\kappa = \kappa^+$

CH : $2^\omega = \omega^+$

10.15 Remark: Silver's Thm: $\beta \in$ Card singular, $cf(\beta) > \omega$, $s := \{\alpha < (\beta \cap \text{Card}) \mid 2^\alpha = \alpha^+\}$ stationary in β $(\forall c \subseteq \lambda \underset{\uparrow}{\text{ closed}}, \text{unbounded}: s \cap c \neq \emptyset) \Rightarrow 2^\beta = \beta^+$.

$[\forall \gamma<\lambda \text{ with } (\forall \alpha<\gamma\ \exists\beta\in c: \alpha<\beta<\gamma): \gamma\in c]$