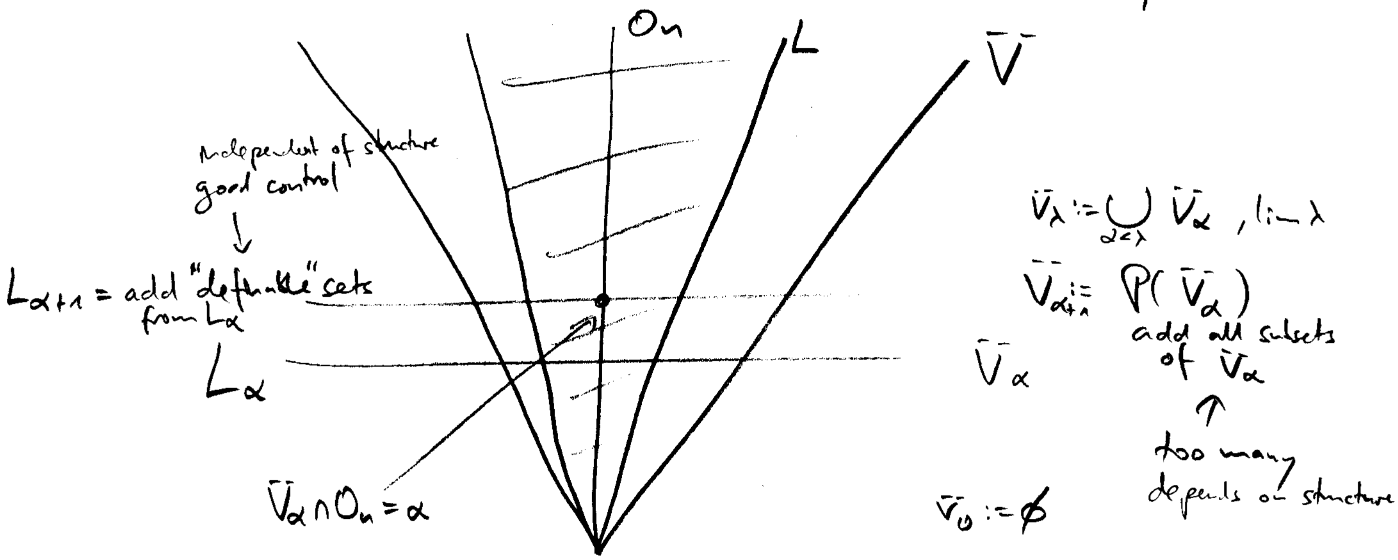
§11 Inner models and $ZF \nvdash \neg CH$

11.1 Motivation: We want to construct a universe L and prove "$cons(ZF) \Longrightarrow cons(ZF + V=L)$". We then show "$ZF + V=L \vdash AC + GCH$". Hence "$cons(ZF) \Rightarrow cons(ZFC+CH)$". In particular $ZF \nvdash \neg CH$. L is constructed as follows:



11.2 Def: (a) A formula φ is $\Delta_0 :\Longleftrightarrow$ there are no unbounded quantifiers in φ

(b) Let $W, A_1, \dots A_n$ be classes. We define $(\varphi)_{W,\vec{A}}$ as follows:

(i) φ atom, i.e. $(x \in y)_{W,\vec{A}} := x \in y$

in $\mathcal{L}$ with $\mathcal{S} = \{A_1, \dots A_n\}$ $\quad (A_i \vec{x})_{W,\vec{A}} := \vec{x} \in A_i$ $(\Leftrightarrow \psi(\vec{x})$, if $A_i = \{\vec{z} \mid \psi(\vec{z})\})$

(ii) $\neg, \wedge, \vee, \rightarrow$ straightforward

(iii) $(\exists x \varphi)_{W,\vec{A}} := \exists x (x \in W \wedge \varphi_{W,\vec{A}})$

$(\forall x \varphi)_{W,\vec{A}} := \forall x (x \in W \rightarrow \varphi_{W,\vec{A}})$

11.3 Thm: Let $T \subseteq \mathcal{L}$ be a set of formulas. $T \vdash \varphi \Rightarrow T_W + (W \neq \emptyset) \vdash \varphi_W$

Proof: 1.) $T = \emptyset$. Let $\mathfrak{S}$ be with $\mathfrak{S} \models (W \neq \emptyset)$, i.e. $\mathfrak{S} \models \exists x \psi(x)$ for $W = \{x \mid \psi(x)\}$.
Let $\mathfrak{S}' := (\{a \in S \mid \mathfrak{S} \models \psi(a)\}, \dots)$. Then $\mathfrak{S}' \models \varphi$, since $\vdash \varphi$. Hence $\mathfrak{S} \models \varphi_W$.

2.) $T \vdash \varphi \xRightarrow{\text{compactness}} \exists T_0 \subseteq_{fin} T : \vdash (\wedge T_0) \rightarrow \varphi \overset{1.)}{\Longrightarrow} W \neq \emptyset \vdash (\wedge (T_0)_W) \rightarrow \varphi_W$
$\Rightarrow (T_0)_W + W \neq \emptyset \vdash \varphi_W$. □

11.4 Lemma: φ Δ_0. Then $\forall a \in W: \varphi_W(a) \Longleftrightarrow \varphi(a)$

Proof: by induction on Def. 11.2(b), idea: only bounded quantifiers involved. □

11.5 Def: Let W be a class. It is an inner model ($IM(W)$), if

$$\forall u \subseteq W\ \exists v : [\underbrace{v \text{ transitive}}_{:\Leftrightarrow\ \forall x,y : [(y \in v \wedge x \in y) \Rightarrow x \in v] \Leftrightarrow \forall y : [y \in v \Rightarrow y \subseteq v]} \wedge u \subseteq v \wedge Def(\langle v, \in \rangle) \subseteq W]$$

where $Def(\langle v,\in\rangle) := \{\{b \in v \mid \underbrace{\langle v, \in \rangle \models \varphi(b, \vec{a})}_{\text{may be defined via some Gödelization}}\} \mid \varphi \text{ formula}, \vec{a} \in v\}$.

11.6 – 11.8

11.9 Thm: $IM(W) \Longleftrightarrow W \text{ transitive} \wedge On \subseteq W \wedge (ZF)_W$

Proof: „$\Leftarrow$" Let $u \subseteq W$, and choose α such that $u \subseteq \underset{\uparrow \text{transitive}}{\bar{V}_\alpha \cap W} =: v$. $ZF \vdash (\bar{V}_\alpha \in \bar{V}) \overset{11.3}{\Longrightarrow} (ZF)_W \vdash ((\bar{V}_\alpha)_W \in W)$, Hence $V_\alpha \subseteq v$. Check also $Def \subseteq W$.

„$\Rightarrow$" W trans.: Let $y \in W$. Hence $u_y := \{y\} \subseteq W$

$\overset{IM}{\Longrightarrow} \exists v$ trans., $y \in v$ (i.e. $y \subseteq v$), $y \subseteq v \overset{11.6}{\subseteq} Def(\langle v, \in \rangle) \subseteq W$

$On \subseteq W$: $On \cap W$ trans. $\overset{[\dots]}{\Longrightarrow} \underbrace{On \cap W \in On}_{\uparrow \text{ can be excluded using } IM(W).} \vee On \cap W = On$

$(ZF)_M$: Check $(Ext)_W$, i.e.

$$(\forall u,v\ \underbrace{(\forall x \in u : x \in v \wedge \forall x \in v : x \in u) \to u = v}_{=: \varphi(u,v)})_W$$

$\Leftrightarrow \forall u,v \in W : \varphi_W(u,v)$.

Now $\varphi = \varphi_W$ since φ is Δ_0, use Lemma 11.4.

And $\forall u,v\ \varphi(u,v)$ by (Ext).

Check also the other axioms ... □

11.7 Def: We put

$L_0 := \emptyset$, $\bar{V}_0 := \emptyset$

$L_{\alpha+1} := Def(\langle L_\alpha, \in \rangle)$, $\bar{V}_{\alpha+1} := P(\bar{V}_\alpha)$

$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$, $\bar{V}_\lambda := \bigcup_{\alpha<\lambda} \bar{V}_\alpha$ lim λ

$L := \bigcup_{\alpha \in On} L_\alpha$, $\bar{V} = \underset{\uparrow\ (\text{Thm})}{\bigcup_{\alpha \in On}} \bar{V}_\alpha$.

11.8 Remark: $\alpha < \omega \Rightarrow \bar{V}_\alpha = L_\alpha$

But $L_{\omega+1} \neq \bar{V}_{\omega+1}$ since $\bar{V}_{\omega+1} = P(\bar{V}_\omega)$ uncountable whereas $L_{\omega+1}$ countable

11.6 Lemma: v transitive $\Rightarrow v \subseteq Def(\langle v, \in \rangle)$.

Proof: Let $b \in v$. Then $b = \{a \in v \mid \langle v, \in \rangle \models (x \in y)[a,b]\} \in Def(\langle v, \in \rangle)$. □

11.9 Thm: (a) We have $L_\alpha \subseteq L_{\alpha+1} \subseteq \dots$, $L_\alpha \in L_{\alpha+1} \in \dots$, L_α transitive, $On \cap L_\alpha = \alpha$.

(b) $IM(L)$. In particular $On \subseteq L$ and $(ZF)_L$.

(c) $(\bar{V} = L)_L$

(d) cons ZF $\Longrightarrow$ cons$(ZF + \bar{V} = L)$

Proof: (a) 11.6 for $L_\alpha \subseteq L_{\alpha+1}$, L_α trans: $y \in \underset{\subseteq P(L_\alpha)}{L_{\alpha+1}} \Rightarrow y \subseteq L_\alpha \subseteq L_{\alpha+1}$.

(b) Let $u \subseteq L$. Put $F(x) := \min\{\alpha \in On \mid x \in L_\alpha\}$ for $x \in u$.

$\overset{(Repl)}{\Longrightarrow} F''u \in \bar{V} \Rightarrow \beta := \sup F''u \in \bar{V} \Rightarrow u \subseteq L_\beta =: v$

and $Def(\langle v, \in \rangle) = L_{\beta+1} \subseteq L$.

(c) $(\bar{V} = L)_L \Leftrightarrow \bar{V}_L = L_L \underset{\uparrow}{\Leftrightarrow} L = L_{On \cap L}$ But $On \cap L \overset{(b)\&11.9}{=} On$ & $L_{On} = L$.

$\left[L_L = \left(\bigcup_{\alpha \in On} L_\alpha\right)_L = \bigcup_{\alpha \in On_L} (L_\alpha)_L = \bigcup_{\alpha \in On \cap L} L_\alpha = L_{On \cap L}\right]$

(d) Assume $\neg$ Cons$(ZF + \bar{V} = L)$, i.e. $ZF + \bar{V} = L \vdash \bot$, $\bot := \varphi \wedge \neg\varphi$.

$\overset{11.3}{\Longrightarrow} (ZF)_L + (\bar{V} = L)_L \vdash \underbrace{\bot_L}_{"\bot}$. Now $ZF \vdash (ZF)_L + (\bar{V} = L)_L$ by (b)&(c)

↯

□

11.10 Thm: $ZF + \bar{V} = L \vdash AC$, even $(GC) :\Leftrightarrow \exists \varphi$ formula well-ordering on $\bar{V}$

In particular, cons(ZF) $\Rightarrow$ Cons(ZFC).

Proof: May define a well-ordering on L:

For $a, b \in Def(\langle x, \in \rangle)$ put $a <'_x b :\Leftrightarrow$

$\exists \varphi$ formula $\exists p \in x^{<\omega} [a = Def(\langle x, \in \rangle, \varphi, p) \wedge \forall \psi$ formula $\forall q \in x^{<\omega}$:

$Def(\langle x, \in \rangle, \psi, q) = b \longrightarrow [\varphi \underset{\uparrow}{<} \psi \vee (\varphi = \psi \wedge p < q)]]]$

formula inductively defined

Δ_0 formula

And $a <''_x b :\Leftrightarrow [a \in x \wedge b \notin x] \vee [a \in x \wedge b \in x \wedge a \underset{\uparrow}{<_x} b] \vee [a \in Def(\langle x, \in \rangle) \setminus x$
$\wedge b \in Def(\langle x, \in \rangle) \setminus x$
$\wedge a <'_x b]$

recursively

and $O(x, <_x) := <''_x$.

Then $<_0 := \emptyset$, $<_{\alpha+1} := O(L_\alpha, <_\alpha)$, $<_\lambda := \bigcup_{\alpha < \lambda} <_\alpha$, λ lim, $<_L := \bigcup_{\alpha \in On} <_\alpha$

Now $GC \vdash AC$: For $u \in \bar{V} = L$ with $\forall x \in u, x \neq \emptyset$: $F: u \longrightarrow \bigcup u$, $x \longmapsto <^L$-min. of x

and $F = F \cap (u \times \bigcup u) \in \bar{V}$

□

11.11 Thm: $ZF + \check{V} = L \vdash GCH$ $(\forall \alpha \in Card \setminus \omega : \overline{\overline{\mathcal{P}(\alpha)}} = \alpha^+)$

In particular $cons\ ZF \Rightarrow cons(ZF + CH)$

Hence $ZFC \nvdash \neg CH$.

Proof: 1.) Need to show: $\mathcal{P}(\alpha) \subseteq L_{\alpha^+}$ for $\alpha \in Card \setminus \omega$.

Indeed, then $\alpha^+ \le \overline{\overline{\mathcal{P}(\alpha)}} \le \overline{\overline{L_{\alpha^+}}} \le \alpha^+$.

$\overline{\overline{L_\alpha}} \le \overline{\overline{\alpha}} \cdot \omega$: Ind($\alpha$). $\alpha \ge 1$.

$\alpha = 1$. $L_1 = Def(\langle \emptyset, \in \rangle) \le \omega$

$\alpha \mapsto \alpha+1$. $L_{\alpha+1} = Def(\langle L_\alpha, \in \rangle)$. $Fml \times L_\alpha^{<\omega} \longrightarrow L_{\alpha+1}$ surj.

$\langle \varphi, \vec{p} \rangle \longmapsto Def(\langle L_\alpha, \in \rangle, \varphi, \vec{p})$

$$\Rightarrow \overline{\overline{L_{\alpha+1}}} \le \overline{\overline{Fml}} \cdot \overline{\overline{L_\alpha^{<\omega}}} \underset{I.Hyp.}{\le} \omega \cdot \overline{\overline{\omega^{<\omega}}} \cdot \overline{\overline{\alpha}} \le \omega \overline{\overline{\alpha}}$$

$$\lim \alpha.\ \overline{\overline{L_\alpha}} = \overline{\overline{\bigcup_{\beta<\alpha} L_\beta}} \le \overline{\overline{\alpha}} \cdot \sup_{\beta<\alpha} \underbrace{\overline{\overline{L_\beta}}}_{\le \omega \cdot \overline{\overline{\beta}}} \le \overline{\overline{\alpha}} \cdot \omega$$

2) So, let $x \subseteq \alpha$. Find $\gamma > \alpha$ with $x \subseteq \alpha \subseteq \gamma \subseteq L_\gamma \Rightarrow x \in L_{\gamma^+}$.

One can show: $L_{\gamma^+} \models ZF^-$ (:= ZF without [illegible] (Power)).

By Löwenheim-Skolem, find substructure $\langle Y, \in \rangle \prec \langle L_{\gamma^+}, \in \rangle$ with $\alpha+1 \subseteq Y$, $x \in Y$, $\overline{\overline{Y}} \le \alpha$.

May then find $\pi : \langle Y, \in \rangle \longrightarrow \langle L_\delta, \in \rangle$ bij. with $\overline{\overline{\delta}} = \overline{\overline{L_\delta}} = \overline{\overline{Y}} \le \alpha$ (i.e. $\delta < \alpha^+$) and $\pi_{|\alpha+1} = id_{|\alpha+1}$ (i.e. $\pi(x) = x$)

Thus $x = \pi(x) \in L_\delta \subseteq L_{\alpha^+}$. □

11.12 Remark: (a) $\diamondsuit$: There is a sequence $\langle S_\alpha, \alpha < \omega_1 \rangle$ with

- $\forall \alpha < \omega_1 : S_\alpha \subseteq \alpha$
- $A \subseteq \omega_1 \Rightarrow \{\alpha < \omega_1 \mid S_\alpha = A \cap \alpha\} \subseteq \omega_1$ stationary (see 10.15) i.e. "dense"

This means: $\langle S_\alpha, \alpha < \omega_1 \rangle$ approximates every $A \subseteq \omega_1$ quite well.
Thus, ω_1 has all information about $\mathcal{P}(\omega_1)$ as power sets are not def.

(b) $ZF + \check{V} = L \vdash \diamondsuit$, $\diamondsuit \vdash CH$

The diamond axiom is an important axiom for other independence results.