§12 Forcing and ZFC $\nvdash$ CH

(an embedded, self-contained talk)

12.1 Aim: Show ~~that~~ (CH): $\overline{\overline{\mathcal{P}(\mathbb{N})}} = \omega_1$ ($\aleph_0, \aleph_1, \aleph_2$; $\omega_0, \omega_1, \omega_2, \ldots$) ; $\mathcal{P}(\mathbb{N}) \cong 2^\omega$; $\omega_0 \cong \mathbb{N}$

is independent from ZFC, i.e.

(1) Cons(ZFC) $\Rightarrow$ Cons(ZFC + CH) [Gödel 1938]

(2) Cons(ZFC) $\Rightarrow$ Cons(ZFC + $\neg$CH) [Cohen 1963]

Solves Hilbert's problem No. 1

Idea: What is a set? If M satisfies the axioms of ZFC, then x set $\Leftrightarrow$ $x \in M$. Write $\bar{V}$ for universe.

Forcing: Add many elements to M such that $\mathcal{P}(\mathbb{N})$ explodes. Control this via generic filter $G \rightsquigarrow M[G]$ extension of M with $M[G] \models 2^\omega \geq \omega_2$.

Idea for (1): Use inner models, i.e. construct an intrinsic model of $\bar{V}$. Then ZFC + $\bar{V} = L$ + CH.

So, forcing needs to destroy $\bar{V} = L$.

12.2 ZFC: (1) Hilbert's 2nd Problem: Are the Peano axioms of number theory free from contradictions?

(2) Russell: $R := \{x \mid x \notin x\}$. Then $R \in R \Leftrightarrow R \notin R$.

$\rightarrow$ R shouldn't be a set.

$\overset{(1)\&(2)}{\rightsquigarrow}$ find axioms for set theory!

ZFC: (Ext) $\forall x \forall y\, [\forall z\, (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$

defines $\bar{V}$ recursively

($\exists \emptyset$) $\emptyset \in \bar{V}$ ($\bar{V}$ "universe"/"all class")

(Pair) $\forall x, y$ $\{x, y\} \in \bar{V}$

(Union) $\forall x$ $\bigcup x := \bigcup_{y \in x} y \in \bar{V}$ (8.3) $\left[\begin{array}{l} x = \{\{0\}, \{1,2\}, \{1,3\}\} \\ \bigcup x = \{0,1,2,3\} \end{array}\right]$

(Sep) $A \cap x \in \bar{V}$ for all classes $A = \{x \mid \varphi(x, \vec{y})\}$

(8.1) $\left[\begin{array}{l} \bullet\ \varphi := (a \in b), \text{ or } \varphi := (a = b) \\ \bullet\ \varphi := (\neg\psi),\ \varphi := (\psi_1 \wedge \psi_2),\ \varphi := (\psi_1 \vee \psi_2), \\ \qquad \varphi := (\psi_1 \rightarrow \psi_1) \\ \bullet\ \varphi := (\forall x\, \psi(x)),\ \varphi := (\exists x\, \psi(x)) \end{array}\right.$

(Repl) F function $\subseteq \bar{V}^2$ $\overset{(8.8)}{\Longrightarrow}$ $\operatorname{ran} F =: F''x \in \bar{V}$ for all $x \in \bar{V}$

(Power) $\forall x \ \ \mathcal{P}(x) := \{z \mid z \subseteq x\} \in \bar{V}$

(Min) $A \neq \emptyset \overset{(8.10)}{\Longrightarrow} \exists x$ $\in$-min. in A for all classes A

$[\forall y \in x : y \notin A \ \ \& \ x \in A]$ (avoids $x \in x$)

(Inf) $\omega \in \bar{V}$

$[\omega := \{ \|\langle U, R\rangle\| \mid \langle U,R\rangle$ finite linear ordering$\} \subseteq On$

all strong well-ordering

$\ulcorner \langle U, R\rangle$ strong well-ordering:

(9.1) • U class, $R \subseteq \bar{V}^2$ ("relation")

• $A \neq \emptyset \Rightarrow \exists x$ R-min. in A for all classes A

$((y,x) \in R \Rightarrow y \notin A)$

• $\forall x: \ \{y \mid (y,x) \in R\} \in \bar{V}$

• $\forall x,y,z \in U$: $(x,x) \notin R$, $(x,y),(y,z) \in R \Rightarrow (x,z) \in R$,

$(x,y) \in R \vee (y,x) \in R \vee x = y$

U transitive: $\forall y \in U \ \forall x \in y : x \in U$

(9.2) Mostowski collapse: $\langle U, R\rangle$ str. well-ord.

$\Longrightarrow \exists V$ transitive: $\langle U,R\rangle \xrightarrow{\cong} \langle V, \in\rangle$

(9.3) $\bar{V} \not\ni On := \{V \mid V$ arises from Mostowski collapses of well-orderings$\}$

$\omega := \{V \mid \langle V, \in\rangle \cong \langle U,R\rangle$ well-ord. & $\langle U, R^{-1}\rangle$ well-ord.$\}$

$A \cap U \neq \emptyset$ has R-min. & R-max. el.

$\rightsquigarrow$ "finite"

(9.3) $\llcorner$ $\alpha < \beta$ on $On :\Leftrightarrow \alpha \in \beta$

(10.5) (AC) $\forall u \ \exists r : \langle u, r\rangle$ well-ordering

$(\Longleftrightarrow \forall x$ with $(\forall t \in x : t \neq \emptyset) : \exists f : u \to \bigcup u$ with $f(x) \in x$

$x \mapsto$

choice function$)$

(

11.3 Cardinals: • We know $u \not\sim \mathcal{P}(u)$, i.e. $\exists f: u \to \mathcal{P}(u)$ injective, but no bijective

• $u \sim v :\iff \exists f: u \to v$ bijective $(u, v \in \bar{V})$

• $\bar{\bar{u}} := \min \{ |\langle u, r\rangle| \mid \langle u, r\rangle \text{ well ordering} \} \sim u$ ↑ Mostowski

Card $:= \{\alpha \in On \mid \bar{\bar{\alpha}} = \alpha\} = \{\aleph_\alpha \mid \alpha \in On\}$, $\aleph_0 = \omega (= \mathbb{N})$

$[\omega+1: 0 < 1 < \dots < \underset{\omega}{\overset{\shortparallel}{\infty}}$, $\omega+1 \sim \omega$: $a \mapsto a+1$ ($a<\omega$), $a \mapsto 0$ ($a = \omega$)

• CH: $2^\omega (= \mathcal{P}(\omega)) = \min \text{Card} \setminus (\omega+1) = \omega_1$

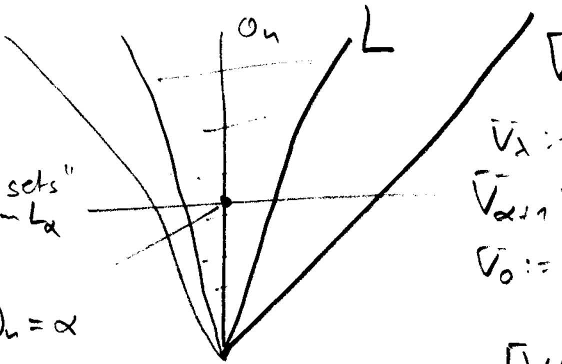
11.4 Inner models: •

$L := \bigcup_{\alpha \in On} L_\alpha$

$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$, lim λ

$L_{\alpha+1} :=$ add "definable sets" from L_α

$L_0 := \emptyset$



$\bar{V} (= \bigcup_{\alpha \in On} \bar{V}_\alpha)$

$\bar{V}_\lambda := \bigcup_{\alpha<\lambda} \bar{V}_\alpha$, lim λ

$\bar{V}_{\alpha+1} := \mathcal{P}(\bar{V}_\alpha)$

$\bar{V}_0 := \emptyset$

$\bar{V}_\alpha \cap On = \alpha$

$[\forall \beta < \lambda \; \exists \gamma < \lambda: \beta < \gamma]$

(11.2) φ formula, $W = \{x \mid \psi(x)\}$ class. Define φ_W recursively:

• $\varphi = (x \in y)$: $\varphi_W = (x \in y)$, $\varphi = (x = y)$: $\varphi_W = (x = y)$

• $\varphi = \neg\psi$, $\varphi = (\psi_1 \overset{\wedge}{\to} \psi_2) \rightsquigarrow \varphi_W = (\neg \psi_W) \mid \varphi_W = (\psi_{1W} \overset{\wedge}{\to} \psi_{2W})$

restrict universe to W ⟶ • $\varphi = (\underset{\forall}{\exists} x\, \psi(x))$, $\varphi_W = (\underset{\forall}{\exists} x \in W: \psi_W(x))$

(11.9) • W is an inner model $:\iff W$ transitive $\wedge$ $On \subseteq W$ $\wedge$ $(ZF)_W$

(11.7) • $L_{\alpha+1} := \{ \{ b \in L_\alpha \mid \langle L_\alpha, \in \rangle \models \varphi(b, \vec{a}) \} \mid \varphi \text{ formula}, \vec{a} \in V \}$

(11.10) • L is an inner model satisfying $(\bar{V} = L)_L$.

• Cons $T :\overset{(4.5)}{\iff} \exists M \models T \iff T \not\vdash \bot$ where $\bot := \psi \wedge \neg\psi$

(11.11 & 11.12) • Gödel 30s: • Cons ZF $\Rightarrow$ Cons$(ZF + \bar{V} = L)$ [since $ZF \vdash (ZF)_L + (\bar{V} = L)_L$]

• $ZF + \bar{V} = L \vdash AC$ [choose well-ordering on L]

• $ZF + \bar{V} = L \vdash CH$ $[\mathcal{P}(\omega) \subseteq L_{\omega_1} \sim \omega_1]$

$L_{\omega_1} \models ZF - (\text{Power})$ [...] counting on definable sets

(4.4) • $ZF + \bar{V} = L \vdash \lozenge$, $\lozenge \vdash CH$

$M \models \varphi$: find model for φ

• elements $x \in \bar{V}$ are interpreted as $x \in M$

• $M \models (x \in y)$ if $x \in y$, $M \models (x = y)$ likewise

• $M \models \neg\psi$ if $M \not\models \psi$, $M \models (\varphi_1 \overset{\wedge}{\to} \varphi_2)$ if $M \models \varphi_1 \overset{\wedge}{\to} M \models \varphi_2$

• $M \models \underset{\forall}{\exists} x\, \psi(x)$ if $\underset{\forall}{\exists} x \in M$ $M \models \psi(x)$

(5.1) $M \vdash \varphi$: inductive definition of logical calculus ("proof")

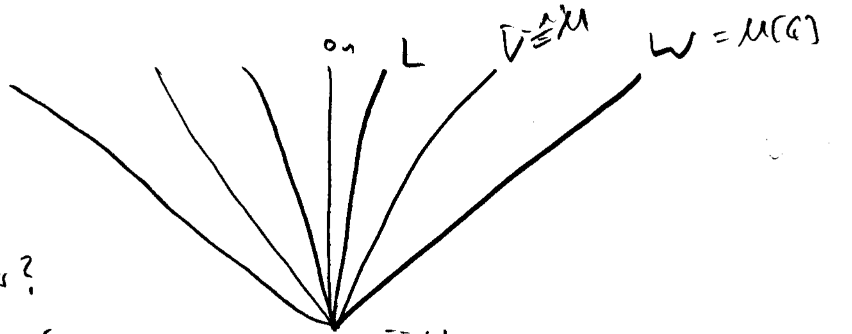
$\varphi \in M \Rightarrow M \vdash \varphi$

$M \vdash \psi$, $M \vdash (\psi \to \varphi) \Rightarrow M \vdash \varphi$

etc.

(5.2/5.3) then $M \vdash \varphi \iff M \models \varphi$

11.5 Idea: • Now, want to construct "outer model" with "$\bar{V} \neq L$"



why outer?

Assume W was an inner model with $\bar{V} \neq L$

$\exists F \vdash IM(W) + \neg(\bar{V}=L)_W \overset{11.3}{\Longleftrightarrow} \exists F_L \vdash (IM(W))_L + ((\bar{V}\neq L)_W)_L$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{IM(L)} \qquad \overset{\shortparallel}{(\bar{V}\neq L)_L}$ ↯ (11.10)

- How to realize? View $\bar{V}$ as a small $\mathcal{M}$ in larger $\bar{V}$. And construct W as extension $\mathcal{M}[G]$ of $\mathcal{M}$
- control this extension via a filter G which is basically a partial order $\mathbb{P}$ on $\mathcal{M}$, defined as something like "$x \leq y :\Longleftrightarrow x \supseteq y$"

This way, may control which "subsets" of $\mathcal{M}$ are actually viewed as sets in $\mathcal{M}[G]$ ⤳ essence of set theory: which classes are allowed to be sets and which aren't?

From the perspective of $\mathcal{M}[G]$ (i.e. $\mathcal{M}[G] \models T$):

x is a set $\Longleftrightarrow$ $x \in \mathcal{M}[G]$

and $x \sim y \Longleftrightarrow \exists f: x \to y$ bijective & $f \in \mathcal{M}[G]$

In the sequel $Fn(I, J)$ will play a crucial role in the sense of "$\exists f: I \twoheadrightarrow J$ surjective". This will yield functions $f: \kappa \times \omega \to \{0,1\}$ and hence many subsets $f_i: \omega \to \{0,1\}$, $i \in \kappa$ of ω. Then $\mathcal{P}(\omega) \geq \kappa$.

11.6 Partial orderings:

- $\mathbb{P}$ partial ordering $:\Leftrightarrow \exists \leq \subseteq \mathbb{P}\times\mathbb{P}$ with $p\leq q\leq r \Rightarrow p\leq r$ and $p\leq p$
- $p \parallel q :\Leftrightarrow \exists r\in\mathbb{P}: r\leq p \wedge r\leq q$ "joint extension"
  $\neg(p\parallel q)$: 
- $D\subseteq\mathbb{P}$ dense $:\Leftrightarrow \forall p\in\mathbb{P}\,\exists q\in D: q\leq p$
- $G\subseteq\mathbb{P}$ filter $:\Leftrightarrow \forall p,q\in G: p\parallel q$ & $\forall p,q\in\mathbb{P}\,(p\in G, p\leq q \Rightarrow q\in G)$
- Let M be transitive, $(ZF-(Power))_M$, $\mathbb{P}\in M$ p.o.
  "large filter" $\Rightarrow$ G $\mathbb{P}$-generic$/M$ $:\Leftrightarrow G\subseteq\mathbb{P}$ filter, $\forall D\in M$ with $D\subseteq\mathbb{P}$ dense have $G\cap D\neq\emptyset$
- $a\in\mathbb{P}$ atom $:\Leftrightarrow \forall p,q\leq a: p\parallel q$ (may not split a in disjoint p and q)
- $\mathbb{P}\in M$ without atoms, G $\mathbb{P}$-gen.$/M$ $\Rightarrow G\notin M$
  [$D:=\mathbb{P}\setminus G\subseteq\mathbb{P}$ dense: For $p\in\mathbb{P}$ ex. $q_0,q_1\leq p$, $\neg(q_0\parallel q_1)$
  Then $\exists q_i\notin G$ since in G all elements compatible $(p\parallel q) \Rightarrow q_i\in D$]

11.7 Construction of $M[G]$:

- Given $M,\mathbb{P},G$ as above.
- Def $M^{\mathbb{P}}$: For $\tau\in M$ such that all of its elements are of the form $\langle\sigma,p\rangle\in\tau$ for some $\sigma\in M^{\mathbb{P}}$, $p\in\mathbb{P}$, define τ_G such that $\sigma_G\in\tau_G$ holds: $\tau_G := \{\sigma_G \mid \exists p\in G: \underset{\{\{\sigma\},\{\sigma,p\}\}}{\langle\sigma,p\rangle}\in\tau\}$
- $M[G] := \{\tau_G \mid \underbrace{\tau\in M \text{ with } \forall x\in\tau\,\exists\sigma,p: x=\langle\sigma,p\rangle, \sigma\in M^{\mathbb{P}}, p\in\mathbb{P}}_{:\Leftrightarrow\, \tau\in M^{\mathbb{P}}}\}$
  generic extension of M by G
- Have: (i) $M\subseteq M[G]$, (ii) $G\in M[G]$, (iii) $M[G]$ transitive, (iv) $On\cap M = On\cap M[G]$

[Proof: (i) $x\in M$. $\check{x} := \{\langle\check{y},1_{\mathbb{P}}\rangle \mid y\in x\}$ where $1_{\mathbb{P}}\in G$ max. el. of $\mathbb{P}$ (exists WLOG)
$x=\check{x}_G\in M[G]$

(ii) $\Gamma := \{\langle\check{p},p\rangle \mid p\in\mathbb{P}\}\in M$, $G \underset{[\dots]}{=} \Gamma_G\in M[G]$

(iii) $\underset{\sigma_G}{a}\in \underset{\tau_G}{b}\in M[G]$, $\sigma\in M$ since M transitive $\Rightarrow \sigma_G\in M[G]$
└

12.8 The forcing relation:

- For $M, \mathbb{P}$, as before, M countable, $\vec{\tau} \in M^{\mathbb{P}}$ and a formula φ define:

$$p \Vdash^M_{\mathbb{P}} \varphi[\vec{\tau}] :\iff \forall G\ \mathbb{P}\text{-gen.}/M: (p \in G \to M[G] \models \varphi[\vec{\tau}_G])$$

- Define $p \Vdash^*_{\mathbb{P}} \varphi[\vec{\tau}]$ recursively on formulas

Begin the hardest

$$p \Vdash^*_{\mathbb{P}} (a \in b)[\tau_1, \tau_2] :\iff \{q \leq p \mid \exists \langle \pi, s\rangle \in \tau_2,\ q \leq s,\ q \in C(\tau_2,\tau_1) \cap C(\tau_1,\tau_2)\} \text{ dense under } p$$

$$(\text{i.e. } \forall r \leq p\ \exists q \in \{\ldots\}: q \leq r)$$

$$p \in C(\tau_1,\tau_2) :\iff \forall \langle \pi_1, s_1\rangle \in \tau_1: \{q \leq p \mid q \leq s_1 \Rightarrow \exists \langle \pi_2, s_2\rangle \in \tau_2 \text{ with } q \leq s_2,\ q \in C(\pi_1,\pi_2) \cap C(\pi_2,\pi_1)\} \text{ dense under } p$$

$$p \Vdash^*_{\mathbb{P}} (a = b)[\tau_1,\tau_2] :\iff p \in C(\tau_1,\tau_2) \cap C(\tau_2,\tau_1)$$

$$\varphi = \varphi_1 \overset{\wedge}{\to} \varphi_2,\quad \varphi = \overset{\exists}{\forall} x\, \varphi_1(x) \ \ldots$$

Miracle: $\Vdash^M_{\mathbb{P}}$ quantifies over objects outside of M and speaks about $M[G] \models$ – for $M[G] \notin M$. However, $\Vdash^*_{\mathbb{P}}$ is defined in M!

- Have $p \Vdash^M_{\mathbb{P}} \varphi[\vec{\tau}] \iff (p \Vdash^*_{\mathbb{P}} \varphi[\vec{\tau}])_M$ (Compare with §5)

[Induction on formulas]

12.9 Forcing Theorem: Let M be transitive with $(ZF^-)_M$ −(Power), $\mathbb{P} \in M$ be a partial order, M be countable, G $\mathbb{P}$-gen./M. Then

$$M[G] \models \varphi[\vec{\tau}_G] \iff \exists p \in G\ \ p \Vdash^M_{\mathbb{P}} \varphi[\vec{\tau}]$$

Proof: "$\Leftarrow$" Def. of $\Vdash^M_{\mathbb{P}}$

"$\Rightarrow$" $\varphi = (a \in b)[\tau_1,\tau_2]$. Construct dense set $D \subseteq \mathbb{P}$ with the help of $C(\tau_1,\tau_2)$, then $G \cap D \neq \emptyset$ [...] $\Rightarrow$ find $p \in G$.

Ind(φ).

12.10 Thm: In the situation above, we have $(ZF - (\text{Power}))_{M[G]}$, $(\text{Power})_M \Rightarrow (\text{Power})_{M[G]}$, $(AC)_M \Rightarrow (AC)_{M[G]}$

11.11 <u>The partial order $Fn(I,J)$:</u>

- Let M be transitive, countable, $(ZFC)^M$.
- For $I,J \neq \emptyset$, define $Fn(I,J)$ on functions p with $\operatorname{dom} p \subseteq I$, $\operatorname{ran} p \subseteq J$, p finite (i.e. $\operatorname{dom} p$ finite) via $p \leq q :\Leftrightarrow p \supseteq q$
- The maximal element is $\emptyset$.
- If $|I| = \infty$, $\overline{\overline{J}} \geq 2$: $Fn(I,J)$ has no atoms

  $\Big[$ For $p \in Fn(I,J)$ $\exists p_0, p_1 \leq p$ with $\neg(p_0 \| p_1)$:
  $\operatorname{dom} p \neq I \Rightarrow \exists i \in I \setminus \operatorname{dom} p$. Let $j_0 \neq j_1 \in J$. $p_k := p \cup \{\langle j_k, i\rangle\}$, $k=0,1$ $\Big]$
- For $\mathbb{P} = Fn(\omega,2)$ and G $\mathbb{P}$-gen./M have $(ZFC + V \neq L)^{M[G]}$, i.e. cons $ZF \Rightarrow$ cons $(ZFC + V \neq L)$

  $\ulcorner$ $\underset{11.7}{G \in M[G]}$, $\underset{11.6}{G \notin M}$ $\Rightarrow$ $G \in M[G] \setminus L_{M[G]}$, $(V \neq L)^{M[G]} = (M[G] \neq L_{M[G]})$

  $\uparrow$ $(G \notin L_M = L_{On \cap M} = L_{On \cap M[G]} = L_{M[G]})$

  $\llcorner$ A: $ZFC + V \neq L \vdash \bot \Rightarrow \underbrace{(ZFC + V \neq L)^{M[G]}}_{\text{holds true}} \vdash \bot$ ↯
- For $|I| = \infty$, $\mathbb{P} := Fn(I,J) \in M$, G $\mathbb{P}$-gen./M have $\bigcup G : I \twoheadrightarrow J$

  $\Big[$ $\operatorname{dom} \bigcup G = I$: For $i \in I$, $D_i := \{p \in \mathbb{P} \mid i \in \operatorname{dom} p\} \subseteq \mathbb{P}$ dense $\Rightarrow \exists p \in D_i \cap G$
  (for $q \in \mathbb{P}$, $i \notin \operatorname{dom} q$: $p := q \cup \{\langle j, i\rangle\}$)
  $\operatorname{ran} \bigcup G = J$: for $j \in J$, $D_j := \{p \mid j \in \operatorname{ran} p\} \subseteq \mathbb{P}$ dense $\Big]$
- For $J = 2$, $\bigcup G = \chi_x$, $x \subseteq \omega$, i.e. $\bigcup G$ defines elements in $\mathcal{P}(\omega)$.
  Have $\bigcup G \in M[G]$, $\bigcup G \notin M$. Thus $\bigcup G$ adds elements to $\mathcal{P}(\omega)$ that were unknown to M $\rightsquigarrow$ Explosion.

11.12 <u>Lemma:</u> For $\mathbb{P} := Fn(\kappa \times \omega, 2)$, G $\mathbb{P}$-gen./M have $M[G] \models 2^\omega \geq \overline{\overline{\kappa}}$

$\ulcorner$ $M[G] \ni \bigcup G : \kappa \times \omega \twoheadrightarrow 2$. Put $F_i : \omega \to 2$, $n \mapsto \bigcup G(i,n)$ for $i < \kappa$.
Then $i \neq j \Rightarrow F_i \neq F_j$ $\llbracket D_{ij} := \{p \in \mathbb{P} \mid \exists n < \omega : p(i,n) \neq p(j,n)\} \overset{\in M}{\subseteq} \mathbb{P}$ dense
$\Rightarrow \exists p \in G \cap D_{ij}$
$\Rightarrow \underbrace{F_i(n) = p(i,n)}_{p \in G} \neq p(j,n) = F_j(n)$

$(F_i \notin M)$

$\llcorner$ Then $\kappa \overset{\text{in } M[G]}{\sim} \{F_i \mid i < \kappa\} \subseteq (\mathcal{P}(\omega))^{M[G]}$

$\ulcorner$ $q \in \mathbb{P} \setminus D_{ij}$. Find $m \in \omega$ with $\langle i, m\rangle \notin \operatorname{dom} q$ & $\langle j, m\rangle \notin \operatorname{dom} q$
$\underset{\leq q}{p} := q \cup \{\langle 0, \langle i,m\rangle\rangle, \langle 1, \langle j,m\rangle\rangle\} \in D_{ij}$
$\rrbracket$

11.13 <u>The ccc</u>:

- $\mathbb{P}$ partial order. $A \subseteq \mathbb{P}$ antichain $:\iff \forall p_0, p_1 \in A: \neg(p_0 \| p_1)$
- $\mathbb{P}$ satisfies ccc $:\iff \forall A \subseteq \mathbb{P}$ antichain: $\overline{\overline{A}} < \omega_1$ (i.e. A countable)
- $(\mathbb{P}\text{ ccc})_M \Rightarrow \forall G$ $\mathbb{P}$-gen./M: $Card_M = Card_{M[G]}$
- $\overline{\overline{J}} \le \omega$, $I \ne \emptyset \to Fn(I,J)$ ccc

11.14 <u>Thm (Cohen 1963)</u>: $\quad$ cons ZFC $\Rightarrow$ cons (ZFC $+ \neg$CH)

Hence, CH independent from ZFC.

<u>Proof</u>: cons ZFC $\Rightarrow$ Cons (ZFC $+$ M countable $+$ M transitive $+ (ZFC)_M$) for an additional predicate symbol M in $\mathcal{L}$.

$\mathbb{P} := Fn((\omega_2)_M \times \omega, 2)$, G $\mathbb{P}$-gen./M, $\kappa := (\omega_2)_M \overset{11.13}{=} (\omega_2)_{M[G]}$

$\overset{11.12}{\Longrightarrow} M[G] \models 2^\omega \ge \overline{\overline{\kappa}} = \omega_2$

$\llcorner$

11.15 <u>Remarks</u>: (a) May realize $M[G] \models 2^\omega = \kappa$ for any $\kappa \in Card_M$ with $(\kappa = \kappa^\omega)_M$

Thus cons (ZF) $\Rightarrow$ cons ZFC $+ 2^\omega = \aleph_\alpha$ for all $cf(\aleph_\alpha) > \omega$

for instance $\alpha = \{1, 2, 3, 4, \_\}$

(b) There is also forcing without ccc.

(c) Solovay: (L) Any set of real numbers is Lebesgue measurable is consistent with ZFC.

(d) without forcing: cons ZFC $\Rightarrow$ cons(ZFC $-$ (Repl) $+ \neg$(Repl))

Cons(ZFC $-$ (Inf) $+ \neg$(Inf))

Cons (ZFC $-$ (Power) $+ \neg$(Power))

(e) 2 July : no lecture (Fri: Coll.)

9 July : Forcing talk by Philipp Lücke

16 July : Operator algebras & forcing, mainly on Farah's article on outer automorphisms of the Calkin algebra