UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 1

Due: Friday, April 20, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 0 (10* points)**.** Make yourself familiar with Matlab. If you do not yet have a copy of Matlab, you can get it from here:

`https://unisb.asknet.de/cgi-bin/product/P10011574`

In particular, you should try to generate random matrices and calculate and plot their eigenvalues. If you hand in some histograms similar to the ones shown in the lecture you can earn some bonus points.

Problem 1 (10 + 10 points)**.** Prove Theorem 0.13 from the lecture, that is, show the following:

(1) (i) For all $k \geq 1$ the Catalan numbers C_k satisfy the recursion relation

$$C_k = \sum_{l=0}^{k-1} C_l C_{k-l-1}.$$

(ii) The Catalan numbers are uniquely determined by the above recursion and the initial value $C_0 = 1$.

(2) The semicircle distribution μ_W is a probability measure, i.e.,

$$\frac{1}{2\pi} \int_{-2}^{2} \sqrt{4 - x^2}\, \mathrm{d}x = 1,$$

and its moments are given by

$$\frac{1}{2\pi} \int_{-2}^{2} x^n \sqrt{4 - x^2}\, \mathrm{d}x = \begin{cases} 0, & n \text{ odd}, \\ C_k, & n = 2k \text{ even}. \end{cases}$$

Please turn the page.

Problem 2 (10 points)**.** Let

$$\Omega_N = \left\{ A = \frac{1}{\sqrt{N}} (a_{ij})_{i,j=1}^N \;\middle|\; A = A^* \text{ and } a_{ij} \in \{-1, 1\} \right\}.$$

Recall that

$$\#\Omega_N = 2^{N(N+1)/2}.$$

In class, we saw that the expectation of $\operatorname{tr}(A^2)$ is 1, i.e.,

$$E[\operatorname{tr}(A^2)] = \frac{\sum_{A \in \Omega_N} \operatorname{tr}(A^2)}{\#\Omega_N} = 1.$$

Do a similar calculation for the expectation of $\operatorname{tr}(A^4)$, i.e., calculate

$$E[\operatorname{tr}(A^4)] = \frac{\sum_{A \in \Omega_N} \operatorname{tr}(A^4)}{\#\Omega_N}$$

as a function of N.
(Note that the calculation for $\operatorname{tr}(A^2)$ was actually quite simple because we have $\operatorname{tr}(A^2) = 1$ for each $A \in \Omega_N$; in the case of $\operatorname{tr}(A^4)$ not all A will give the same contribution.)