UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 2

Due: Friday, April 27, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 3 (5 points)**.** Let A be a symmetric matrix in $M_N(\mathbb{R})$ and consider its empirical spectral measure $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A. Prove that for any $z \in \mathbb{C}_+$ the Stieltjes transform g_{μ_N} of μ_N is given by

$$g_{\mu_N}(z) = \operatorname{tr}(A - zI)^{-1}.$$

Problem 4 (20 points)**.** Prove Proposition 1.6 from the lecture: Let μ be a measure on $\mathbb{R}$ with finite total mass $\mu(\mathbb{R})$ and denote by g_μ its Stieltjes transform. Show the following:

(i) For any $z \in \mathbb{C}_+$, $|g_\mu(z)| \leq \frac{\mu(\mathbb{R})}{\operatorname{Im}(z)}$.

(ii) g_μ is analytic over $\mathbb{C}_+$.

(iii) $\operatorname{Im}(g_\mu(z)) > 0$ for all $z \in \mathbb{C}_+$.

(iv) If $\operatorname{supp} \mu \subset \mathbb{R}_+$ then $\operatorname{Im}(z g_\mu(z)) \geq 0$ for all $z \in \mathbb{C}_+$.

(v) $\lim_{y\to\infty} iy g_\mu(iy) = -\mu(\mathbb{R})$.

Please turn the page.

Problem 5 (10 + 10 points)**.** Denote by μ the semicircle distribution.

(i) Prove, by using the Catalan recurrence relation

$$C_k = \sum_{l=0}^{k-1} C_l C_{k-l-1} \quad (k \geq 1), \qquad C_0 = 1, \tag{1}$$

that the Stieltjes transform g_μ of μ satisfies the equation

$$g_\mu(z) = -\frac{1}{z} - \frac{1}{z}\big(g_\mu(z)\big)^2$$

for all $z \in \mathbb{C}_+$ with $|z| > 2$.

Hint: Use the Cauchy product formula for power series.

(ii) Deduce that for all $z \in \mathbb{C}_+$,

$$g_\mu(z) = \frac{-z + \sqrt{z^2 - 4}}{2},$$

and use this to get back the original measure μ via the inversion formula.

Problem 6 (10* points)**.** In this problem we revisit, inspired by the previous problem, Problem 1 from last week. (Note that you cannot obtain additional points with this exercise if you already solved Problem 1 (1) correctly.)

We want to show that the recurrence relation (1) has the Catalan numbers as (only) solution. In order to do this, employ the generating function

$$f(z) = \sum_{k=0}^{\infty} C_k z^k$$

and proceed as follows:

(i) Show that (1) implies $f(z) = 1 + z f(z)^2$.

(ii) Show that f is a power series representation for

$$\frac{1 - \sqrt{1 - 4z}}{2z}.$$

(iii) Show that

$$C_k = \frac{1}{k+1}\binom{2k}{k}.$$