UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Prof. Dr. Roland Speicher
Dr. Marwa Banna
M.Sc. Ricardo Schnur

Random Matrices

Summer term 2018

Assignment 3

Due: Friday, May 4, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 7 (10 points)**.** Let $Z, Z_1, \ldots, Z_r$ be independent standard complex Gaussian random variables with mean 0 and $\mathbb{E}\left[|Z_i|^2\right] = 1$ for $i = 1, \ldots, n$.

(i) Show that
$$\mathbb{E}\left[Z_{i_1} \cdots Z_{i_r} \overline{Z_{j_1}} \cdots \overline{Z_{j_r}}\right] = \#\{\sigma \in S_r \mid i = j \circ \sigma\}.$$

(ii) Show that
$$\mathbb{E}\left[Z^n \overline{Z}^m\right] = \begin{cases} 0, & m \neq n, \\ n!, & m = n. \end{cases}$$

Problem 8 (10 points)**.** Let $A = (a_{ij})_{i,j=1}^N$ be a Gaussian (GUE) random matrix with entries $a_{ij} = x_{ij} + iy_{ij}$ and with the normalization $\mathbb{E}\left[|a_{ij}|^2\right] = \frac{1}{N}$. Consider the N^2 random vector
$$(x_{11}, \ldots, x_{NN}, x_{12}, \ldots x_{1N}, \ldots, x_{N-1,N}, y_{12}, \ldots, y_{N-1,N})$$
and show that it has density
$$c \exp\left(-N \frac{\operatorname{Tr}(A^2)}{2}\right) \mathrm{d}A,$$
where c is a constant and
$$\mathrm{d}A = \left(\prod_{i=1}^N \mathrm{d}x_{ii}\right)\left(\prod_{i<j} \mathrm{d}x_{ij} y_{ij}\right)$$
is the Lebesgue measure on $\mathbb{R}^{N^2}$.

Please turn the page.

Problem 9 (20 + 5* points)**.** Let μ be a probability measure on $\mathbb{R}$ with moments

$$m_k = \int_{\mathbb{R}} x^k \, \mathrm{d}\mu(x) \qquad (k \in \mathbb{N}_0)$$

and let φ_μ be its characteristic function given by

$$\varphi_\mu \colon \mathbb{R} \to \mathbb{C},\, t \mapsto \int_{\mathbb{R}} e^{itx} \, \mathrm{d}\mu(x).$$

(i) Calculate the Taylor series expansion of φ_μ at $t = 0$.

(ii) Show that the following conditions are equivalent:

  (a) φ_μ is analytic over $\mathbb{R}$.

  (b) φ_μ is analytic at 0.

  (c) $\limsup_{k\to\infty} \left(\frac{1}{k!} |m_k|\right)^{\frac{1}{k}} < \infty$

(iii) Deduce that if one of the equivalent conditions in (ii) is satisfied, μ is characterized by its moments.

(iv) Show that the real Gaussian measure is determined by its moments.

(v*) Use the Carleman condition to show directly that the real Gaussian measure is determined by its moments.

Hint: The characteristic function φ_μ uniquely determines the measure μ.