UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Prof. Dr. Roland Speicher
Dr. Marwa Banna
M.Sc. Ricardo Schnur

Random Matrices

Summer term 2018

Assignment 4

Due: Friday, May 11, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 10 (20 points)**.** (For this exercise you have two weeks, so until May 18!)
Produce histograms for various random matrix ensembles.

(i) Produce histograms for the averaged situation: Average over 1000 realizations for the eigenvalue distribution of an $N \times N$ Gaussian random matrix (or, alternatively, with ± 1 entries) and compare this with one random realization for $N = 5, 50, 500, 1000$.

(ii) Check via histograms that Wigner's semicircle law is insensitive to the common distribution of the entries as long as those are independent, that is, compare typical realizations for $N = 100$ and $N = 3000$ for different distributions of the entries:

± 1, Gaussian, uniform distribution on the interval $[-1, +1]$

(iii) Check what happens when we give up the constraint that the entries are centered, consider for example the uniform distribution on $[0, 2]$.

(iv) Check whether the semicircle law is sensitive to what happens on the diagonal of the matrix. Choose one distribution (e.g. Gaussian) for the off-diagonal elements and another distribution for the elements on the diagonal.

Extreme case: Put the diagonal entries all equal to zero

(v) Try to see what happens when we take a distribution for the entries which does not have a finite second moment, such as the Cauchy distribution.

Problem 11 (10 points)**.** Let $A = (a_{ij})_{i,j}, B = (b_{ij})_{i,j} \in M_n(\mathbb{R})$ be $n \times n$-matrices and consider a $n \times 1$-random vector

$$x = (x_1, \ldots, x_n)^T,$$

where the x_i are independently identically distributed random variables with $\mathbb{E}[x_i] = 0$ for all $i = 1, \ldots, n$.

(i) Prove that

$$\mathbb{E}\left[(x^T A x)(x^T B x)\right] = \left(\mathbb{E}\left[x_1^4\right] - 3\mathbb{E}\left[x_1^2\right]^2\right)\sum_{i=1}^n a_{ii}b_{ii} \\ + \mathbb{E}\left[x_1^2\right]^2\left(\mathrm{Tr}(A)\,\mathrm{Tr}(B) + \mathrm{Tr}(AB) + \mathrm{Tr}\left(AB^T\right)\right).$$

(ii) Suppose that $\mathbb{E}\left[x_i^2\right] = 1$ for all $i = 1, \ldots, n$ and that A is symmetric. Prove that

$$\mathbb{E}\left[\left(x^T A x - \mathbb{E}\left[x^T A x\right]\right)^2\right] = \left(\mathbb{E}\left[x_1^4\right] - 3\right)\sum_{i=1}^n a_{ii}^2 + 2\,\mathrm{Tr}\left(A^2\right).$$

(iii) Suppose that $x_i \sim \mathcal{N}(0,1)$ for all $i = 1, \ldots, n$. What is the value of $\mathrm{Var}\left(x^T A x\right)$?

Recall: The variance of a random variable Y is defined as $\mathrm{Var}(Y) = \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right]$.

Problem 12 (10 points)**.** The problems with being determined by moments and whether convergence in moments implies weak convergence are mainly coming from the behaviour of our probability measures around infinity. If we restrict everything to a compact interval, then the main statements follow quite easily by relying on the Weierstrass theorem for approximating continuous functions by polynomials.

In the following let $I = [-K, K]$ be a fixed compact interval in $\mathbb{R}$.

(i) Assume that μ is a probability measure on $\mathbb{R}$ which has its support in I (that is, $\mu(I) = 1$). Show that all moments of μ are finite and that μ is determined by its moments.

(ii) Consider in addition a sequence of probability measures $(\mu_n)_n$, such that $\mu_n(I) = 1$ for all n. Show that the following are equivalent:

- μ_n converges weakly to μ.
- The moments of μ_n converge to the corresponding moments of μ.

Problem 13 (10 points)**.** In class we have seen that the m-th moment of a Wigner matrix is asymptotically counted by the number of partitions $\sigma \in \mathcal{P}(m)$, for which the corresponding graph $\mathcal{G}_\sigma$ is a tree. Then the corresponding walk $i_1 \to i_2 \to \cdots \to i_m \to i_1$ (where $\ker i = \sigma$) uses each edge exactly twice, in opposite directions. Assign to such a σ a pairing by opening/closing a pair when an edge is used for the first/second time in the corresponding walk.

(i) Show that this map gives a bijection between the $\sigma \in \mathcal{P}(m)$ for which $\mathcal{G}_\sigma$ is a tree and the non-crossing pairings $\pi \in \mathcal{NC}_2(m)$.

(ii) Is there a relation between σ and $\gamma\pi$ under this bijection?