UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 5

Due: Friday, May 18, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 14 (20 points)**.** Recall that the spectral norm of a matrix A is defined as

$$\|A\| = \max\left\{\sqrt{\lambda};\ \lambda \text{ is an eigenvalue of } A^*A\right\}.$$

For a vector $a \in \mathbb{C}^n$, we denote by $\|\cdot\|_2$ the Euclidean norm given by

$$\|a\|_2 = \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}}.$$

Recall further that if A is Hermitian with Eigenvalues $\lambda_1, \ldots, \lambda_n$, then by the spectral theorem there is a unitary matrix U such that

$$A = U\Lambda U^*,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

(i) Show that if A is real symmetric then

$$\|A\| = \max\{|\lambda|\,;\ \lambda \text{ is an eigenvalue of } A\}.$$

(ii) Show that $\|A\| \leq \max\{\|Ax\|_2\,;\ \|x\|_2 \leq 1\}$.

(iii) Let $a \in \mathbb{C}^n$ and suppose that A is the rank-one matrix given by $A = aa^*$. Show that a is an eigenvector of A and specify its associated eigenvalue.

(iv) Show that $\|A\| = \sup\{|y^*Ax|\,;\ \|x\|_2 \leq 1, \|y\|_2 \leq 1\}$.
Hint: Write $|y^*Ax|^2 = x^*A^*yy^*Ax$.

(v) Deduce that for all $x, y \in \mathbb{C}^n$, $|y^*Ax| \leq \|A\|\,\|x\|_2\,\|y\|_2$.

(vi) Let A be positive semi-definite. Show that for any matrix B,

$$|\mathrm{Tr}(AB)| \leq \|B\|\,\mathrm{Tr}(A).$$

Please turn the page.

Problem 15 (20 points)**.** The Hilbert-Schmidt norm (L^2-norm) of a matrix $A = (a_{ij})_{i,j=1}^n$ is defined by

$$\|A\|_2 = (\mathrm{Tr}\,(A^*A))^{\frac{1}{2}} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

(i) Let A, B, U be $n \times n$-matrices. Prove that:

(1) $|Tr(AB)| \leq \|A\|_2 \cdots \|B\|_2$

(2) If U is unitary then $\|AU\|_2 = \|UA\|_2 = \|A\|_2$.

(3) If B is normal then $\max\{\|AB\|_2, \|BA\|_2\} \leq \|B\|\,\|A\|_2$.

(ii) Let A be selfadjoint and let $G_A(z) = (A - zI)^{-1}$ for $z \in \mathbb{C}_+$. Prove that

$$\|G_A(z)\| \leq \frac{1}{\mathrm{Im}(z)}.$$