UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 6

Due: Friday, May 25, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 16 (10 points)**.** (Matrix identities) Let $u, v, w \in \mathbb{C}^n$ and let $A, B \in M_n(\mathbb{C})$ be invertible.

(i) Show that for each $m \in \mathbb{N}_0$,

$$A^{-1} = \sum_{k=0}^{m} B^{-1} \left[(B-A)B^{-1}\right]^k + A^{-1} \left[(B-A)B^{-1}\right]^{m+1}.$$

(ii) Show that

$$(I + wv^*)^{-1} = I - \frac{wv^*}{1 + v^* w}.$$

(iii) Suppose that the rank-one perturbation $A + uv^*$ is invertible. Deduce the Sherman-Morrison identity

$$(A + uv^*)^{-1} = A^{-1} - \frac{A^{-1} u v^* A^{-1}}{1 + v^* A^{-1} u}.$$

(iv) Deduce

$$\operatorname{Tr}(A + uv^*)^{-1} = \operatorname{Tr}\left(A^{-1}\right) - \frac{v^* \left(A^{-1}\right)^2 u}{1 + v^* A^{-1} u}$$

and

$$(A + uv^*)^{-1} uv^* = A^{-1} \frac{uv^*}{1 + v^* A^{-1} u}.$$

Problem 17 (20 points)**.** Let $u \in \mathbb{C}^n$ and $A, B \in M_n(\mathbb{C})$ with A selfadjoint. The aim of this exercise is to prove that for all $z \in \mathbb{C}_+$,

$$\left|\operatorname{Tr}\left[(A + uu^* - zI)^{-1} B\right] - \operatorname{Tr}\left[(A - zI)^{-1} B\right]\right| \leq \frac{\|B\|}{\operatorname{Im} z}. \qquad (\star)$$

(i) Prove, using the Sherman-Morrison identity, that for all $z \in \mathbb{C}_+$,

$$\operatorname{Tr}\left[(A + uu^* - zI)^{-1} B\right] - \operatorname{Tr}\left[(A - zI)^{-1} B\right] = -\frac{u^*(A - zI)^{-1} B (A - zI)^{-1} u}{1 + u^*(A - zI)^{-1} u}.$$

(ii) Prove that for all $z \in \mathbb{C}_+$,

$$\operatorname{Im}\left(1 + u^*(A - zI)^{-1}u\right) = \operatorname{Im}(z) \sum_{i=1}^{n} \frac{|u^* v_i|^2}{|\lambda_i - z|^2},$$

where the λ_i are the eigenvalues of A and the v_i are the associated eigenvectors with $\|v_i\|_2 = 1$ for each $i = 1, \ldots, n$.

(iii) Recall that $\|B\| = \sup_{\|u\|,\|v\| \leq 1} |u^* B v|$ to prove that for all $z \in \mathbb{C}_+$,

$$\left|u^*(A - zI)^{-1}B(A - zI)^{-1}u\right| \leq \|B\| \cdot \left\|(A - zI)^{-1}u\right\|^2.$$

(iv) Deduce that $(\star)$ holds.

Problem 18 (20 points)**.** Let $\{X_{ij};\, i, j \in \mathbb{N}\}$ be a family of random variables such that $X_{ij} = \overline{X_{ji}}$. Let X_N be the selfadjoint $N \times N$-matrix defined by

$$X_N = \left(\frac{1}{\sqrt{N}} X_{ij}\right)_{i,j=1}^{N}.$$

(i) We shall see in this part that diagonal entries do not contribute to the limit. Consider the matrix $X_N^{(0)}$ obtained from X_N by replacing the diagonal entries by zero. Prove that for any $z \in \mathbb{C}_+$,

$$\left|g_{X_N}(z) - g_{X_N^{(0)}}(z)\right| \xrightarrow{N \to \infty} 0.$$

(ii) In this part, we show a possible way of truncating the matrix entries and considering bounded variables.

(1) Let Y_N and A_N be selfadjoint $N \times N$-matrices and let $\tilde{Y}_N = Y_N + A_N$ be a perturbation of Y_N. Prove that for any $z \in \mathbb{C}_+$,

$$\left|\operatorname{Tr}\left(\frac{1}{\sqrt{N}} Y_N - zI\right)^{-1} - \operatorname{Tr}\left(\frac{1}{\sqrt{N}} \tilde{Y}_N - zI\right)^{-1}\right| \leq \frac{1}{(\operatorname{Im} z)^2} \sqrt{\operatorname{Tr} A_N^2}.$$

(2) Let $\{\tilde{X}_{ij};\, i, j \in \mathbb{N}\}$ be the family of random variables defined by

$$\tilde{X}_{ij} = X_{ij} 1_{\{|X_{ij}| < \sigma\sqrt{N}\}}$$

for some $\sigma > 0$ and let

$$\tilde{X}_N = \left(\frac{1}{\sqrt{N}} \tilde{X}_{ij}\right)_{i,j=1}^{N}.$$

Suppose that for all $\tau > 0$,

$$L_N(\tau) = \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E}\left[X_{ij}\right]^2 1_{\{|X_{ij}| \geq \tau\sqrt{N}\}} \xrightarrow{N \to \infty} 0.$$

Show that X_N and $\tilde{X}_N$ have the same limiting distribution.

Problem 19 (10 points)**.** Let A be the mapping given by

$$A\colon \mathbb{R}^{\frac{N(N+1)}{2}} \to M_N(\mathbb{R}),\ (x_{ij})_{1\le j\le i\le N} \mapsto [A(x)_{ij}]_{i,j=1}^{N}$$

with

$$A(x)_{ij} = \frac{1}{\sqrt{N}}\begin{cases} x_{ij}, & i \ge j, \\ x_{ji}, & i < j. \end{cases}$$

For any $z \in \mathbb{C}_+$, let $G = G_z$ be given by $G_z(x) = (A(x) - zI)^{-1}$.

(i) Let $k, l \in \{1, \dots, N\}$ wit $k \neq l$. Show that

$$\left[\frac{\partial G}{\partial x_{kk}}(x)\right]_{ij} = -\frac{1}{\sqrt{N}} G_{ik}G_{kj}$$

and

$$\left[\frac{\partial G}{\partial x_{kl}}(x)\right]_{ij} = -\frac{1}{\sqrt{N}}\left(G_{ik}G_{lj} + G_{il}G_{kj}\right).$$

(ii) Define

$$g_z\colon \mathbb{R}^{\frac{N(N+1)}{2}} \to \mathbb{C},\ x \mapsto \frac{1}{N}\operatorname{Tr} G_z(x).$$

Prove that, for any choice of indices,

$$\begin{aligned} \left|\partial_{x_{i_1,j_1}} g_z(x)\right| &\le \frac{2}{(\operatorname{Im} z)^2 N^{\frac{3}{2}}}, \\ \left|\partial_{x_{i_2,j_2}}\partial_{x_{i_1,j_1}} g_z(x)\right| &\le \frac{4}{(\operatorname{Im} z)^3 N^{2}}, \\ \left|\partial_{x_{i_3,j_3}}\partial_{x_{i_2,j_2}}\partial_{x_{i_1,j_1}} g_z(x)\right| &\le \frac{3\cdot 2^{\frac{5}{2}}}{(\operatorname{Im} z)^4 N^{\frac{5}{2}}}. \end{aligned}$$