**UNIVERSITÄT DES SAARLANDES**
**FACHRICHTUNG MATHEMATIK**
Prof. Dr. Roland Speicher
Dr. Marwa Banna
M.Sc. Ricardo Schnur

# Random Matrices

Summer term 2018

## Assignment 7

**Due:** Friday, June 1, 2018, before the lecture

Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

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**Problem 20** (15 points). (i) In the first part of this exercise we shall prove the weak law of large numbers: let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with common mean $\mu = \mathbb{E}[X_i]$ and variance $\sigma^2 = \mathbb{V}\text{ar}[X_i]$. Let

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Show that $\forall t > 0$,

$$\mathbb{P}[|S_n - \mu| \geq t] \xrightarrow{n\to\infty} 0.$$

(ii) In this part we shall prove the strong law of large numbers under the assumption that the fourth moment is finite.

(1) Let $X_1, \ldots, X_n$ be independent random variables with common mean $\mu = \mathbb{E}[X_i]$ and define $S_n$ as above. Suppose that

$$M_4 = \sup_{1\leq i\leq n} \mathbb{E}\left[(X_i - \mu)^4\right] < \infty.$$

Show that

$$\mathbb{E}\left|\sum_{i=1}^{n}(X_i - \mu)\right|^4 \leq 3n^2 M_4$$

and deduce that $\forall t > 0$,

$$\mathbb{P}[|S_n - \mu| \geq t] \leq \frac{3M_4}{n^2 t^4}.$$

(2) Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with common mean $\mu = \mathbb{E}[X_i]$ and such that $\mathbb{E}[X_i^4] < \infty$. Define $S_N$ as above and show that

$$S_n \xrightarrow{n\to\infty} \mu \qquad \text{almost surely.}$$

**Problem 21** (20 points). (i) Let $b > 0, \sigma \geq 0$ and let $X$ be a real random variable with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] \leq \sigma^2$. Show the following:

(1) If $|X| \leq b$ almost surely then for all $\lambda \geq 0$,

$$M_X(\lambda) \leq \exp\left(\frac{\sigma^2}{b^2}\left(e^{\lambda b} - \lambda b - 1\right)\right).$$

(2) If for all $k \geq 3$,

$$\mathbb{E}\left[X^k\right] \leq \frac{1}{2}k!\sigma^2 b^{k-2}$$

then for all $\lambda \in \left[0, \frac{1}{b}\right]$,

$$M_X(\lambda) \leq \exp\left(\frac{\lambda^2\sigma^2}{2(1-\lambda b)}\right).$$

(ii) Let $b, \sigma_1, \ldots, \sigma_n > 0$ and let $X_1, \ldots, X_n$ be independent *centered* random variables such that $\mathbb{E}[X_i^2] \leq \sigma_i^2$ for all $i = 1, \ldots, n$ and

$$\mathbb{E}\left[X_i^k\right] \leq \frac{1}{2}k!\sigma_i^2 b^{k-2}$$

for all $k \geq 3$. Show that for every $t > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2(\sigma^2 + tb)}\right)$$

where $\sigma^2 = \sum_{i=1}^n \sigma_i^2$.

(iii) Let $b, \sigma_1, \ldots, \sigma_n > 0$ and let $X_1, \ldots, X_n$ be independent centered random variables such that $\mathbb{E}[X_i^2] \leq \sigma_i^2$ and $|X_i| \leq b$ almost surely for all $i = 1, \ldots, n$. Show that for every $t > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| \geq t\right) \leq 2\exp\left(-\frac{\sigma^2}{b^2}H\left(\frac{tb}{\sigma^2}\right)\right),$$

where $H(x) = (1+x)\log(1+x) - x$.

**Problem 22** (5 points). Let $d$ be a fixed narural number and $(u_i)_{i\in\mathbb{N}}$ be a sequence of centered random vectors in $\mathbb{R}^d$ with covariance matrix $\Sigma = \mathbb{E}\left[u_i u_i^T\right]$. Let $\mathcal{X}_n = (u_1, \ldots, u_n)$ be the $d \times n$-matrix whose columns are the $u_i$.

(i) Show that almost surely,

$$\frac{1}{n}\mathcal{X}_n\mathcal{X}_n^T = \frac{1}{n}\sum_{i=1}^{n} u_i u_i^T \xrightarrow{n\to\infty} \Sigma.$$

(ii) Fix $z \in \mathbb{C}_+$ and consider the map

$$\Phi_z\colon M_N(\mathbb{C}) \to M_N(\mathbb{C}),\ A \mapsto (A - zI)^{-1}.$$

Show that $\Phi_z$ is continuous.

(iii) Let $\mu_n$ and $\mu^\Sigma$ be the empirical spectral measures of $\frac{1}{n}\mathcal{X}_n\mathcal{X}_n^T$ and $\Sigma$, respectively. Show that $\mu_n \xrightarrow{w} \mu^\Sigma$ almost surely as $n \to \infty$.

(iv) Calculate the limiting distribution as $n \to \infty$ for $\frac{1}{n}\mathcal{X}_n\mathcal{X}_n^T$ under the assumption $\Sigma = \sigma^2 I$.

**Problem 23** (20 points)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independently identically distributed standard Gaussian random variables $\mathcal{N}(0,1)$ and let $A$ and $(A_n)_{n\in\mathbb{N}}$ be real symmetric positive semidefinite deterministic matrices. We are interested in studying the asymptotic behavior of $x_n^T A_n x_n$ , where $x_n = (X_1, \ldots, X_n)^T$. Denote by $\mu^{A_n}$ and $\mu^A$ the empirical spectral measures of $A_n$ and $A$ respectively, and assume that $\mu^{A_n} \xrightarrow{w} \mu^A$.

(i) Show that if

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}\setminus[-k,k]} |x| \, \mathrm{d}\mu^{A_n}(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}} |x| \, \mathrm{d}\mu^A(x) < \infty \tag{$\star$}$$

then

$$\frac{1}{n} \operatorname{Tr} A_n \xrightarrow{n\to\infty} \int_{\mathbb{R}} x \, \mathrm{d}\mu^A(x).$$

(ii) Show that $(\star)$ is satisfied if $\sup_{n\in\mathbb{N}} \|A_n\| \le M < \infty$ for some $M > 0$. Deduce that

$$\frac{1}{n}\mathbb{E}\left[x_n^T A_n x_n\right] \xrightarrow{n\to\infty} \int_{\mathbb{R}} x \, \mathrm{d}\mu^A(x).$$

(iii) Let $O_n$ be an orthogonal $n \times n$-matrix. Compute the expectation and covariance matrix of the Gaussian vector $O_n x_n$.

(iv) Deduce that there exist independently identically distributed standard Gaussian random variables $Y_i$ such that

$$\frac{1}{n} x_n^T A_n x_n = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^{A_n} Y_i^2,$$

where the $\lambda_i^{A_n}$ are the eigenvalues of $A_n$.

(v) Show that

$$\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^{n} \lambda_i^{A_n} Z_i\right|^4 \le \frac{K}{n^2},$$

where $Z_i = Y_i^2 - 1$.

(vi) Deduce that almost surely,

$$\frac{1}{n} x_n^T A_n x_n - \mathbb{E}\left[\frac{1}{n} x_n^T A_n x_n\right] \xrightarrow{n\to\infty} 0.$$