UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Random Matrices

Summer term 2018

Assignment 8

Due: Tuesday, June 19, 2018, before the lecture
Hand in your solution at the beginning of the lecture or drop it into letterbox 47.

Problem 24 (20 points)**.** Produce histograms for the averaged eigenvalue distribution of a GUE(N) and compare this with the exact analytic density from class.

(i) First, transform the averaged eigenvalue density

$$p_N(\mu) = \frac{1}{N} K_N(\mu,\mu) = \frac{1}{\sqrt{2\pi}} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{k!} H_k(\mu)^2 e^{-x^2/2}$$

for the unnormalized GUE(N) into the density $q_N(\lambda)$ for the normalized GUE(N) (with second moment normalized to 1).

(ii) Then average over sufficiently many normalized GUE(N), plot their histograms, and compare this to the analytic density $q_N(\lambda)$. Do this at least for $N = 1, 2, 3, 5, 10, 20, 50$.

(iii) Check numerically that q_N converges to the semicircle for $N \to \infty$.

(iv) For comparison, also average over Wigner ensembles with non-Gaussian distribution for the entries for some small N.

Problem 25 (10 points)**.** Let $(X, \mathcal{B}, \nu)$ be a measure space, and let ϕ_i, ψ_j, $1 \le i, j \le N$ be measurable functions such that $\phi_i \psi_j$ is integrable for any i, j. Prove the Cauchy-Binet identity:

$$\int_{X^N} \det\big(\phi_i(x_j)\big)_{i,j=1}^N \cdot \det\big(\psi_i(x_j)\big)_{i,j=1}^N \, d^N\nu(x) = N! \det\left(\int_X \phi_i(x)\psi_j(x)\, d\nu(x)\right)_{i,j=1}^N$$

Hint: Use the Leibniz expansion of the determinant of a square matrix.

Please turn the page.

Problem 26 (10 points)**.** In this exercise we define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2}}$$

and want to show that they are the same polynomials we defined in class and that they satisfy the recursion relation. So, starting from the above definition show the following:

(i) H_n is a monic polynomial of degree n. Furthermore, it is an even function if n is even and an odd function if n is odd.

(ii) The H_n are orthogonal with respect to the Gaussian measure $\mathrm{d}\gamma(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\,\mathrm{d}x$. More precisely, show the following:

$$\int_{\mathbb{R}} H_n(x) H_m(x)\,\mathrm{d}\gamma(x) = \delta_{nm} n!$$

(iii) For any $n \geq 1$,

$$x H_n(x) = H_{n+1}(x) + n H_{n-1}(x).$$

Problem 27 (10 points)**.** Let

$$p(\mu_1, \ldots, \mu_N) = \frac{1}{N!} \det\big(K_N(\mu_i, \mu_j)\big)_{i,j=1}^N$$

be the density for the joint eigenvalue distribution for the unnormalized GUE(N), where K_N is the Hermite kernel

$$K_N(x, y) = \sum_{k=0}^{N-1} \psi_k(x) \psi_k(y).$$

Let f be an integrable function. Then we are interested in the following expectation:

$$\mathbb{E}\left(\prod_{i=1}^N (1 - f(\mu_i))\right) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\mu_1, \mu_2, \ldots, \mu_N) \prod_{i=1}^N (1 - f(\mu_i)) d\mu_1 \cdots d\mu_N$$

(i) Give an interpretation for the quantity $\mathbb{E}\left(\prod_{i=1}^N (1 - f(\mu_i))\right)$ in the case where f is the characteristic function of a subset $I \subset \mathbb{R}$, i.e.,

$$f(\mu) = \begin{cases} 0, & \mu \notin I, \\ 1, & \mu \in I. \end{cases}$$

(ii) Show, by using the Cauchy-Binet identity from Problem 25, that

$$\mathbb{E}\left(\prod_{i=1}^N (1 - f(\mu_i))\right) = \det\left(\delta_{ij} - \int_{-\infty}^{\infty} \psi_i(\mu) \psi_j(\mu) f(\mu)\, d\mu\right)_{i,j=1}^N.$$